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SCORE PROBABILITIES FOR SERVE AND RALLY COMPETITIONS

ILAN ADLER,* *University of California, Berkeley*
SHELDON M. ROSS,** *University of Southern California*

Abstract

We consider serve and rally competitions involving two teams, in which the probability that a team wins a rally depends on which team is serving. We give elementary derivations of the final score probabilities both when the match ends when one of the teams reaches a set number of points, and when there is an additional proviso that the winning team must be ahead by at least two points. We consider models where the winner of a rally receives a point, and also where the winner of a rally receives a point only if that player was the server of the rally. In the latter case we also compute the mean number of rallies. We also determine conditions under which a player would prefer to be the initial server.

Keywords: Point a rally; side out scoring; win by two; mean number of rallies

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1. Introduction

We consider serve and rally competitions involving two players, who we refer to as A and B, and which are such that either the winner of a rally is always given a point or the winner of the rally only earns a point if that player was the server for that rally. For instance, the traditional versions of squash, racquetball, badminton, and volleyball were of the serve and rally format in which the winner of a rally becomes the next server but only earns a point if they were the server in that rally. More recently, however, rules in some of these games have been changed so that the winner of a rally earns a point no matter who served that rally. For instance, international volleyball rules were changed in 1999 to give a point to the winner of a rally. American squash also uses this scoring system.

Assuming that A wins a rally with probability p_a when A is the server and with probability p_b when B is the server and that all the rallies are independent, we derive score probabilities and the mean number of rallies played when the game ends when one of the players has amassed a total of at least n points, possibly subject to a condition such as the winner must always have at least two more points than the loser. In addition, we derive the conditions under which a player would prefer to serve in the beginning of the game. Unless otherwise specified, we assume that player A serves first and that (with the exception of the last section) the winner of a rally is serving in the next one. Throughout the paper we use the notation $q_x = 1 - p_x,\ x = a, b$. Although some of our results have previously appeared in the literature (see [1], [2], [3], and [4]), our methods are new and simpler than ones previously used.

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* Postal address: Department of IEOR, University of California, Berkeley, CA 94720, USA.
Email address: adler@ieor.berkeley.edu
** Postal address: Epstein Department of Industrial and Systems Engineering, University of Southern California, Los Angeles, CA 90089, USA. Email address: smross@usc.edu

2. Point a rally model

Assume that scoring is according to the point a rally model, where the winner of the rally, regardless of who served, receives a point. Say that a new round begins each time A serves. Let B_i denote the number of wins of B in round i and let $S_k = \sum_{i=1}^{k} B_i$. Note that S_k is the number of wins of B at the moment when A has won her kth point. Also, note that $B_i = 0$ with probability p_a, and with probability q_a is distributed as a geometric random variable with parameter p_b.

Let $X_{i,j}$ denote the event that the $(i+j)$th point was won by X and that A has i and B has j points at that time.

We use the convention that $\binom{r}{s} = 0$ if $s > r$.

Proposition 1. *For $i > 0$*

$$\mathrm{P}(A_{i,0}) = p_a^i,$$

$$\mathrm{P}(A_{i,j}) = p_a^i q_b^j \sum_{r=1}^{i} \binom{i}{r}\binom{j-1}{r-1}\left(\frac{q_a p_b}{p_a q_b}\right)^r, \qquad j > 0.$$

Proof. The result for $\mathrm{P}(A_{i,0})$ is immediate, so suppose that $j > 0$. Because A wins his ith point at the end of round i, we have

$$\mathrm{P}(A_{i,j}) = \mathrm{P}(S_i = j).$$

Conditioning on the number of $B_1, \ldots, B_i$ that are positive and then using that the sum of independent and identically distributed geometrics is a negative binomial gives

$$\mathrm{P}(S_i = j) = \sum_{r=1}^{i} \binom{i}{r} q_a^r p_a^{i-r} \binom{j-1}{r-1} p_b^r q_b^{j-r}.$$

Proposition 2. *For $j > 0$*

$$\mathrm{P}(B_{0,j}) = q_a q_b^{j-1},$$

$$\mathrm{P}(B_{i,j}) = q_a q_b^{j-1} p_a^i \left[1 + \sum_{s=1}^{j-1}\sum_{r=1}^{i} \binom{i}{r}\binom{s-1}{r-1}\left(\frac{q_a p_b}{p_a q_b}\right)^r\right], \qquad i > 0.$$

Proof. The result for $\mathrm{P}(B_{0,j})$ is immediate. For $i > 0$, condition on S_i to obtain

$$\begin{aligned}
\mathrm{P}(B_{i,j}) &= \sum_{s=0}^{j-1} \mathrm{P}(B_{i,j} \mid S_i = s)\, \mathrm{P}(S_i = s) \\
&= \sum_{s=0}^{j-1} q_a q_b^{j-1-s}\, \mathrm{P}(S_i = s) \\
&= q_a q_b^{j-1} p_a^i \left[1 + \sum_{s=1}^{j-1}\sum_{r=1}^{i} \binom{i}{r}\binom{s-1}{r-1}\left(\frac{q_a p_b}{p_a q_b}\right)^r\right].
\end{aligned}$$

Suppose that the game ends when either A has a total of n points or B has a total of m points. Let X_A and X_B denote the number of wins of A and of B when the game ends. If $X_A = n$ say that A wins, and if $X_B = m$ say that B wins.

Corollary 1. *We have*

$$\begin{aligned} \mathrm{P}(X_A = n, X_B = j) &= \mathrm{P}(A_{n,j}), \qquad j < m, \\ \mathrm{P}(X_A = i, X_B = m) &= \mathrm{P}(B_{i,m}), \qquad i < n, \end{aligned}$$

$$\mathrm{P}(A \text{ wins}) = \sum_{j=0}^{m-1} \mathrm{P}(A_{n,j}).$$

2.1. When you have to win by at least two points

Suppose now that play continues until one of the players has won at least n points and is leading the other player by at least two points. Since the game will continue forever if $p_a = q_b = 0$, we assume that $p_a + q_b > 0$. Let T_X be the event that the two players are tied with $n-1$ points for each player and that player $X (X = A, B)$ is to serve next. Then we have

$$\begin{aligned} \mathrm{P}(A \text{ wins} \mid T_A) &= p_a^2 + q_a p_b \, \mathrm{P}(A \text{ wins} \mid T_A) + p_a q_a \, \mathrm{P}(A \text{ wins} \mid T_B), \\ \mathrm{P}(A \text{ wins} \mid T_B) &= p_b p_a + q_b p_b \, \mathrm{P}(A \text{ wins} \mid T_A) + p_b q_a \, \mathrm{P}(A \text{ wins} \mid T_B). \end{aligned}$$

Solving gives

$$\mathrm{P}(A \text{ wins} \mid T_A) = \frac{p_a^2}{p_a^2 + q_b^2 + p_a q_b (1 - p_a - q_b)}, \tag{1}$$

$$\mathrm{P}(A \text{ wins} \mid T_B) = \frac{p_a^2 + p_a q_b (1 - p_a - q_b)}{p_a^2 + q_b^2 + p_a q_b (1 - p_a - q_b)}. \tag{2}$$

Now, player A wins either if she scores n points while her opponent scores $n-2$ points or less, or if the score is tied at $n-1$ and she proceeds to eventually win. Therefore,

$$\mathrm{P}(A \text{ wins}) = \mathrm{P}(A \text{ wins} \mid T_A)\, \mathrm{P}(T_A) + \mathrm{P}(A \text{ wins} \mid T_B)\, \mathrm{P}(T_B) + \sum_{i=0}^{n-2} \mathrm{P}(A_{n,i}), \tag{3}$$

which can be computed using (1) and (2), and the results

$$\mathrm{P}(T_A) = \mathrm{P}(A_{n-1,n-1}) = (p_a q_b)^{n-1} \sum_{r=1}^{n-1} \binom{n-1}{r} \binom{n-2}{r-1} \left(\frac{q_a p_b}{p_a q_b} \right)^r \tag{4}$$

and

$$\begin{aligned} \mathrm{P}(T_B) &= \mathrm{P}(B_{n-1,n-1}) \\ &= q_a p_a^{n-1} q_b^{n-2} \left[1 + \sum_{s=1}^{n-2} \sum_{r=1}^{n-1} \binom{n-1}{r} \binom{s-1}{r-1} \left(\frac{q_a p_b}{p_a q_b} \right)^r \right]. \end{aligned} \tag{5}$$

To determine the probabilities of final scores in which the winner ends with more than n points let $T_X(k)$ be the event that the score is tied at k points and that player X is to serve.

Starting with $\mathrm{P}[T_A(n-1)] = \mathrm{P}(T_A)$ and $\mathrm{P}[T_B(n-1)] = \mathrm{P}(T_B)$ (as given in (4) and (5)), we have, for $k \geq n$,

$$\mathrm{P}[T_A(k)] = q_a p_b \,\mathrm{P}[T_A(k-1)] + q_b p_b \,\mathrm{P}[T_B(k-1)],$$
$$\mathrm{P}[T_B(k)] = p_a q_a \,\mathrm{P}[T_A(k-1)] + p_b q_a \,\mathrm{P}[T_B(k-1)],$$

which can be used to calculate (for $k \geq n+1$)

$$\mathrm{P}(X_A = k, X_B = k-2) = p_a^2 \,\mathrm{P}[T_A(k-2)] + p_b p_a \,\mathrm{P}[T_B(k-2)],$$
$$\mathrm{P}(X_A = k-2, X_B = k) = q_a q_b \,\mathrm{P}[T_A(k-2)] + q_b^2 \,\mathrm{P}[T_B(k-2)].$$

Assume now that the players are tied at $n-1$ points apiece and let R_A and R_B denote the number of additional points won, respectively, by A and by B.

Proposition 3. *We have*

$$\mathrm{E}[R_A \mid T_A] = 2\left[\frac{p_a + p_a q_a + q_b p_b + p_a q_b(p_a + q_b - 2)}{p_a^2 + q_b^2 + p_a q_b(1 - p_a - q_b)}\right],$$
$$\mathrm{E}[R_A \mid T_B] = 2\left[\frac{p_b(p_a + 2q_b)}{p_a^2 + q_b^2 + p_a q_b(1 - p_a - q_b)}\right],$$
$$\mathrm{E}[R_B \mid T_A] = 2\left[\frac{q_a(q_b + 2p_a)}{p_a^2 + q_b^2 + p_a q_b(1 - p_a - q_b)}\right],$$
$$\mathrm{E}[R_B \mid T_B] = 2\left[\frac{q_b + p_a q_a + q_b p_b + p_a q_b(p_a + q_b - 2)}{p_a^2 + q_b^2 + p_a q_b(1 - p_a - q_b)}\right].$$

Proof. This follows by solving

$$\mathrm{E}[R_A \mid T_A] = 2p_a^2 + q_a p_b(2 + \mathrm{E}[R_A \mid T_A]) + p_a q_a(2 + \mathrm{E}[R_A \mid T_B]),$$
$$\mathrm{E}[R_A \mid T_B] = 2p_b p_a + q_b p_b(2 + \mathrm{E}[R_A \mid T_A]) + p_b q_a(2 + \mathrm{E}[R_A \mid T_B])$$

and

$$\mathrm{E}[R_B \mid T_A] = 2q_a q_b + q_a p_b(2 + \mathrm{E}[R_B \mid T_A]) + p_a q_a(2 + \mathrm{E}[R_B \mid T_B]),$$
$$\mathrm{E}[R_B \mid T_B] = 2q_b^2 + q_b p_b(2 + \mathrm{E}[R_B \mid T_A]) + p_b q_a(2 + \mathrm{E}[R_B \mid T_B]).$$

Corollary 2. *We have*

$$\mathrm{E}[X_A] = n \sum_{j=0}^{n-2} \mathrm{P}(A_{n,j}) + \sum_{i=1}^{n-2} i\,\mathrm{P}(B_{i,n})$$
$$+ (n - 1 + \mathrm{E}[R_A \mid T_A])\,\mathrm{P}(T_A) + (n - 1 + \mathrm{E}[R_A \mid T_B])\,\mathrm{P}(T_B),$$
$$\mathrm{E}[X_B] = n \sum_{i=0}^{n-2} \mathrm{P}(B_{i,n}) + \sum_{j=1}^{n-2} j\,\mathrm{P}(A_{n,j})$$
$$+ (n - 1 + \mathrm{E}[R_B \mid T_A])\,\mathrm{P}(T_A) + (n - 1 + \mathrm{E}[R_B \mid T_B])\,\mathrm{P}(T_B).$$

3. Side out scoring model

In the side out scoring model a player only wins a point if she wins the rally while on serve. If a player wins a rally that was started by the other player serving then the winner of the rally serves next but no point is earned. To determine the probabilities for the final score, let p_A denote the probability that A wins the next point given that A has just won a point, and let p_B denote the probability that A wins the next point given that B has just won a point. Then we obtain

$$p_A = p_a + q_a p_b p_A,$$

giving that

$$p_A = \frac{p_a}{1 - q_a p_b}.$$

Also,

$$p_B = p_b p_A = \frac{p_a p_b}{1 - q_a p_b}.$$

All probabilities concerning the final score can now be obtained from the corresponding probabilities of Section 2 upon replacing p_a by p_A and p_b by p_B.

Proposition 4. *If W denotes the total number of rallies, then*

$$\mathrm{E}[W] = (1 + \mathrm{E}[X_A] - \mathrm{P}(A \text{ wins}))\frac{1 + q_a}{1 - q_a p_b} + (\mathrm{E}[X_B] - \mathrm{P}(B \text{ wins}))\frac{1 + p_b}{1 - q_a p_b}.$$

If the match ends when one of the players has n points, then $\mathrm{E}[X_A]$, $\mathrm{E}[X_B]$ and $\mathrm{P}(A$ wins$)$ in the preceding are obtainable from Corollary 1 upon replacing p_a by p_A and p_b by p_B. If we add the requirement that the winner must win by at least two points, then $\mathrm{E}[X_A]$, $\mathrm{E}[X_B]$ and $\mathrm{P}(A$ wins$)$ are obtainable from Corollary 2 and (3) upon replacing p_a by p_A and p_b by p_B.

Proof. Let R_a and R_b denote the number of rallies until a point is scored when the initial server is, respectively, A and B. Then,

$$\mathrm{E}[R_a] = 1 + q_a\,\mathrm{E}[R_b]$$

and

$$\mathrm{E}[R_b] = 1 + p_b\,\mathrm{E}[R_a],$$

giving that

$$\mathrm{E}[R_a] = \frac{1 + q_a}{1 - q_a p_b}, \qquad \mathrm{E}[R_b] = \frac{1 + p_b}{1 - q_a p_b}.$$

As before, let X_A and X_B be the number of wins of A and B at the moment the game ends, but assume that the players continue to play for additional points even after the game has been won. Whenever A wins a point, say that the set of rallies that occur until the next point is won constitute an A-batch of rallies. Let V_i denote the number of rallies in the ith A-batch. By Wald's equation, we have

$$\mathrm{E}\left[\sum_{i=1}^{X_A} V_i\right] = \mathrm{E}[X_A]\,\mathrm{E}[R_a].$$

Similarly, whenever B wins a point, say that the set of rallies that occur until the next point is won constitute a B-batch of rallies. Let U_i denote the number of rallies in the ith B-batch, and again use Wald's equation to obtain

$$\mathrm{E}\left[\sum_{i=1}^{X_B} U_i\right] = \mathrm{E}[X_B]\,\mathrm{E}[R_b].$$

Now with F equal to the number of rallies until the first point, and M equal to the number of rallies played after the game ends until the next point is won, we obtain

$$W = F + \sum_{i=1}^{X_A} V_i + \sum_{i=1}^{X_B} U_i - M.$$

Hence,

$$\begin{aligned}\mathrm{E}[W] &= \mathrm{E}[F] + \mathrm{E}[X_A]\,\mathrm{E}[R_a] + \mathrm{E}[X_B]\,\mathrm{E}[R_b] - \mathrm{E}[M \mid A \text{ wins}]\,\mathrm{P}(A \text{ wins}) \\ &\quad - \mathrm{E}[M \mid B \text{ wins}]\,\mathrm{P}(B \text{ wins}) \\ &= (1 + \mathrm{E}[X_A] - \mathrm{P}(A \text{ wins}))\,\mathrm{E}[R_a] + (\mathrm{E}[X_B] - \mathrm{P}(B \text{ wins}))\,\mathrm{E}[R_b].\end{aligned}$$

4. When to choose to serve first

Lemma 1. *In the side out scoring model of Section 3,*

$$\mathrm{P}(A \text{ wins} \mid A \text{ serves first}) \geq \mathrm{P}(A \text{ wins} \mid B \text{ serves first}).$$

Proof. Clearly

$$\begin{aligned}&\mathrm{P}(A \text{ wins} \mid A \text{ serves first}) \\ &\quad = p_a\,\mathrm{P}(A \text{ wins} \mid A \text{ leading 1:0 and } A \text{ to serve}) + q_a\,\mathrm{P}(A \text{ wins} \mid B \text{ serves first}).\end{aligned}$$

But since

$$\mathrm{P}(A \text{ wins} \mid A \text{ leading 1:0 and } A \text{ to serve}) \geq \mathrm{P}(A \text{ wins} \mid A \text{ serves first}),$$

we get that

$$\mathrm{P}(A \text{ wins} \mid A \text{ serves first}) \geq \mathrm{P}(A \text{ wins} \mid B \text{ serves first}).$$

Proposition 5. *In the point a rally model where the winner of the rally earns a point no matter who served,*

$$p_a \geq p_b \quad \Rightarrow \quad \mathrm{P}(A \text{ wins} \mid A \text{ serves first}) \geq \mathrm{P}(A \text{ wins} \mid B \text{ serves first}).$$

Proof. Let $p_a \geq p_b$ and assume, without loss of generality, that $0 < p_b < 1$. Imagine a game using the scheme of Section 3 – that is, one in which a point is only won if the winner of the rally was the server. Suppose in this game that A wins a rally that starts with A serving with probability p_a^*, and that A wins a rally that starts with B serving with probability p_b^*, where

$$p_a^* = \frac{p_a - p_b}{1 - p_b}$$

and

$$p_b^* = \frac{p_b}{p_a}.$$

Then, by the discussion in Section 3, we can convert this to an equivalent game with scoring scheme of Section 2.1 – that is, one where the winner of a rally gains a point no matter who served in that rally – with

$$\begin{aligned} P(A \text{ wins rally when } A \text{ serves}) &= \frac{p_a^*}{1 - q_a^* p_b^*} \\ &= \frac{p_a - p_b}{1 - p_b}\left[1 - \left(\frac{1 - p_a}{1 - p_b}\right)\frac{p_b}{p_a}\right]^{-1} \\ &= \frac{p_a(p_a - p_b)}{p_a(1 - p_b) - (1 - p_a)p_b} \\ &= p_a \end{aligned}$$

and

$$\mathrm{P}(A \text{ wins rally when } B \text{ serves}) = p_b^* \, \mathrm{P}(A \text{ wins rally when } A \text{ serves}) = p_b.$$

The result now follows by Lemma 1.

Remark. Proposition 5 was previously proven in [1] by a much more complicated approach that used Legendre polynomials to express the probability that A wins the game. In fact, Proposition 5 is applicable to other approaches to determine the winner as long as

$$\mathrm{P}(A \text{ wins} \mid A \text{ leading 1:0 and } A \text{ to serve}) \geq \mathrm{P}(A \text{ wins} \mid A \text{ serves first}).$$

5. When the loser serves

Suppose now that the rules are changed so that the loser of a rally is the next to serve. Then, if we change *serve* to *receive*, denote by p_a the probability that A wins a rally when B serves, denote by p_b the probability that A wins a rally when A serves, and assume that A is to receive first, we can directly apply the results of the previous sections to competitions following the loser-serve protocol.

6. Examples

Under USA Racquetball and Racquetball Canada rules, matches are the best of three games with the first two games played to 15 points and a third game, if necessary, played to 11 points. USA Racquetball rules do not require players to win a game by at least a two point margin, while Racquetball Canada rules do require at least a two point margin.

Suppose that $p_a = 0.56$ and $p_b = 0.48$. If the winner of a game is the first to get to 15, the probabilities that A wins in the point a rally (PaR) and in the side out scoring (SoS) models, both when the winning team has to, and when it does not have to, win by at least two points are as follows:

$$\mathrm{P}(A \mid \text{PaR: need not win by 2}) = 0.5922,$$
$$\mathrm{P}(A \mid \text{PaR: must win by 2}) = 0.5919,$$
$$\mathrm{P}(A \mid \text{SoS: need not win by 2}) = 0.6449,$$
$$\mathrm{P}(A \mid \text{SoS: must win by 2}) = 0.6360.$$

If the winner of a game is the first to get to 11, and it is not necessary to win by 2, then the win probabilities for A as well as $\mathrm{E}[W]$, the expected number of rallies, are as follows:

$$\mathrm{P}(A \mid \mathrm{PaR}) = 0.5808,$$
$$\mathrm{E}[W \mid \mathrm{PaR}] = 18.0310,$$
$$\mathrm{P}(A \mid \mathrm{SoS}) = 0.6337,$$
$$\mathrm{E}[W \mid \mathrm{SoS}] = 31.3371.$$

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