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# The equivalence of linear programs and zero-sum games

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**Abstract** In 1951, Dantzig showed the equivalence of linear programming problems and two-person zero-sum games. However, in the description of his reduction from linear programs to zero-sum games, he noted that there was one case in which the reduction does not work. This also led to incomplete proofs of the relationship between the Minimax Theorem of game theory and the Strong Duality Theorem of linear programming. In this note, we fill these gaps.

**Keywords** Linear programming · Zero-sum games · Minimax theorem · Strong duality · Farkas' lemma · Villes' theorem

## 1 Introduction

This note concerns the equivalence between linear programming (LP) problems and zero-sum games. As presented in texts discussing the relationship between two-person zero-sum games and LP (e.g. Luce and Raiffa (1957); Raghavan (1994)), LP problems and zero-sum games are claimed to be equivalent in the sense that one can convert any two-person zero-sum game to an LP problem and vice-versa. The reduction of any zero-sum game to an LP problem is well known to be simple and direct. In addition, this reduction can be applied to prove the Minimax Theorem as a direct corollary of the Strong Duality Theorem of LP. The reduction of an LP problem (and its dual) to a zero-sum game is somewhat more involved. Consider the LP problem:

$$\begin{aligned} \min \quad & c^\mathsf{T} x \\ \text{s.t} \quad & Ax \geq b, \qquad x \geq 0 \end{aligned} \tag{1}$$

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and its dual:

$$\begin{array}{lll} \max & b^{\top}y & \\ \text{s.t} & A^{\top}y \leq c, & y \geq 0 \end{array} \tag{2}$$

where $A \in \mathcal{R}^{m\times d}$, $b, y \in \mathcal{R}^m$, $c, x \in \mathcal{R}^d$.

A zero-sum game whose payoff matrix $P \in \mathcal{R}^{n\times n}$ is skew symmetric (i.e. $P^{\top} = -P$) is called a *symmetric game*. A *solution* for such a game is a vector $z \in \mathcal{R}^n$ satisfying:

$$Pz \geq 0, \quad e^{\top}z = 1, \quad z \geq 0 \ \ \text{(where } e \text{ is a vector of 1's).} \tag{3}$$

The Minimax Theorem, as first introduced in von Neumann (1928), and as applied to a symmetric zero-sum game, states that:

**Theorem 1** (Minimax Theorem—symmetric zero-sum game) *Given a skew-symmetric matrix $P \in \mathcal{R}^{n\times n}$, there exists a vector $z \in \mathcal{R}^n$ satisfying (3).*

It is suggested in Dantzig (1951) to reduce the pair of LP problems (1–2) to a symmetric zero-sum game with the following payoff matrix:

$$P = \begin{pmatrix} 0 & A & -b \\ -A^{\top} & 0 & c \\ b^{\top} & -c^{\top} & 0 \end{pmatrix} \tag{4}$$

We call such a game the *Danzig game* associated with problems (1–2). Applying Theorem 1 to (4), a solution $(\bar{y}^{\top} \ \ \bar{x}^{\top} \ \ \bar{t})^{\top}$ to the corresponding Dantzig game satisfies:

$$\begin{pmatrix} 0 & A & -b \\ -A^{\top} & 0 & c \\ b^{\top} & -c^{\top} & 0 \end{pmatrix} \begin{pmatrix} \bar{y} \\ \bar{x} \\ \bar{t} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \bar{y} \\ \bar{x} \\ \bar{t} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad e^{\top}\bar{y} + e^{\top}\bar{x} + \bar{t} = 1. \tag{5}$$

It is shown in Dantzig (1951) that:

**Theorem 2** (a) $\bar{t}(b^{\top}\bar{y} - c^{\top}\bar{x}) = 0$.

(b) *If $\bar{t} > 0$, then $\bar{x}\bar{t}^{-1}$, $\bar{y}\bar{t}^{-1}$ are optimal solutions to the primal–dual LP problems pair (1–2), respectively.*

(c) *If $b^{\top}\bar{y} - c^{\top}\bar{x} > 0$, then either problem (1) or problem (2) (possibly both) are infeasible (so neither has an optimal solution).*

However, this reduction of an LP problem to a zero-sum game is incomplete. Specifically, it is possible to have a solution $(\bar{y}^{\top} \ \ \bar{x}^{\top} \ \ \bar{t})^{\top}$ to (5) with $\bar{t} = b^{\top}\bar{y} - c^{\top}\bar{x} = 0$ which, by itself, does not resolve the optimality, or lack of optimality, of the pair of LP problems (1–2). Indeed, the first two references to the equivalence of LP problems and zero-sum games, Dantzig (1951) and Gale et al. (1951), explicitly point out this discrepancy. Again, in his LP tome Dantzig (1963), Dantzig states that "[t]he reduction of a linear program to a game depends on finding a solution of a game with $t > 0$.

If $t = 0$ in a solution, it does not necessarily mean that an optimal feasible solution to the linear program does not exist." In addition, the typical proof of the Strong Duality Theorem of LP via the Minimax Theorem (e.g. Raghavan (1994)) uses Theorem 2 together with a version of the well known Farkas' Lemma (see e.g. Dantzig (1963)) which is used to show that there always exists a solution $(\bar{y}^\mathsf{T} \quad \bar{x}^\mathsf{T} \quad \bar{t})^\mathsf{T}$ to (5) where either $\bar{t} > 0$ or $b^\mathsf{T}\bar{y} - c^\mathsf{T}\bar{x} > 0$, and as a consequence of this, one can easily obtain the Strong Duality Theorem. The problem with this approach is that it is well known that the Strong Duality Theorem can be derived directly as a special case of Farkas' Lemma. So in a sense, this approach amounts to a tautology. Over the years, the equivalence of LP problems and zero-sum games seemed to be accepted as folk theory. The goal of this note is to plug this "hole" and to provide a complete treatment of the equivalence between LP problems and zero-sum games.

## 2 Problem setup, and summary of the results

To simplify the presentation, we replace the LP problem with a feasibility problem in the so-called *Standard Form*. Specifically, given $A \in \mathcal{R}^{m\times n}, \quad b \in \mathcal{R}^m$, let

$$X(A, b) = \{x \mid Ax = b,\ x \geq 0\}.$$

We define the *Linear feasibility* (LF) problem as:

$$\text{F}(A, b): \qquad \text{Either find } x \in X(A, b), \text{ or show that } X(A, b) = \emptyset.$$

We assume, without loss of generality, that $A_{\bullet j} \neq 0$ for $j = 1, \ldots, n$, and that $rank(A) = m$.

Let $Y(A, b) = \{y \mid y^\mathsf{T}A \leq 0,\ y^\mathsf{T}b > 0\}$. The following lemma, which is the LF problem's analogue of the Weak Duality Theorem of LP, provides a sufficient condition for the infeasibility of $Ax = b,\ x \geq 0$.

**Lemma 3** *If $Y(A, b) \neq \emptyset$, then $X(A, b) = \emptyset$.*

*Proof* Suppose $\bar{x} \in X(A, b)$ and $\bar{y} \in Y(A, b)$, then

$$0 \geq (\bar{y}^\mathsf{T}A)\bar{x} = \bar{y}^\mathsf{T}(A\bar{x}) = \bar{y}^\mathsf{T}b > 0.$$

a contradiction. □

If (as in Dantzig (1951)—see Theorem 2) by solving an LP problem one means finding an optimal solution if one exists, or else demonstrating that no optimal solution exists, there is no loss of generality in considering the LF problem instead of the LP problem. This is because it is well known that one can present a pair of dual LP problems (such as (1) and (2)) as an LF problem by combining the constraints of the primal and the dual and adding the constraint $b^\mathsf{T}y \geq c^\mathsf{T}x$.

In order to present $F(A, b)$ as a symmetric game, we replace the equation $Ax = b$ with the equivalent two inequalities $Ax \geq b$ and $(-e^\mathsf{T}A)x \geq -e^\mathsf{T}b$. To construct the associated Dantzig game we consider the following LP problem:

$$\begin{aligned} & \min \quad u \\ \text{s.t} \quad & Ax + eu \geq b \\ & (-e^\top A)x + u \geq -e^\top b \\ & x \geq 0, \ u \geq 0 \end{aligned}$$

whose dual is

$$\begin{aligned} & \max \quad b^\top y - (e^\top b)v \\ \text{s.t} \quad & A^\top y - (A^\top e)v \leq 0 \\ & e^\top y + v \leq 1 \\ & y \geq 0, \quad v \geq 0. \end{aligned}$$

Constructing the corresponding Dantzig game, we observe that its set of solutions (see 1, 2, 3, and 4) is:

$$\left\{ \begin{pmatrix} y \\ v \\ x \\ u \\ t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \;\middle|\; \begin{pmatrix} 0 & 0 & A & e & -b \\ 0 & 0 & -e^\top A & 1 & e^\top b \\ -A^\top & A^\top e & 0 & 0 & 0 \\ -e^\top & -1 & 0 & 0 & 1 \\ b^\top & -b^\top e & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ v \\ x \\ u \\ t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \right.$$

$$\left. e^\top y + v + e^\top x + u + t = 1 \right\}$$

where $x \in \mathcal{R}^n$, $y \in \mathcal{R}^m$, and $u, v, t \in \mathcal{R}$.

We refer to a game with the payoff matrix above as the *Dantzig game associated with* $F(A, b)$. Expanding the system of inequalities above, we get:

$$Ax + eu \geq bt, \tag{6a}$$

$$(e^\top A)x - u \leq (e^\top b)t, \tag{6b}$$

$$A^\top(y - ev) \leq 0, \tag{6c}$$

$$e^\top y + v \leq t, \tag{6d}$$

$$b^\top(y - ev) \geq u, \tag{6e}$$

$$x \geq 0, \ u \geq 0, \ y \geq 0, \ v \geq 0, \ t \geq 0 \tag{6f}$$

$$e^\top x + u + e^\top y + v + t = 1. \tag{6g}$$

Thus, we can refer to the set of all the solutions of the Dantzig game associated with $F(A, b)$ as the set of all $(x, u, y, v, t)$ satisfying (6a–6g). We denote this set by $G(A, b)$. In the next proposition we prove the analogue of Theorem 2 for LF problems and specify a condition that must be satisfied whenever $F(A, b)$ is not resolved by solving its associated Dantzig game.

**Proposition 4** *Let* $(\tilde{x}, \tilde{u}, \tilde{y}, \tilde{v}, \tilde{t}) \in G(A, b)$.

(a) *Suppose* $\tilde{t} > 0$,

  (i) *If* $\tilde{u} = 0$, *then* $\tilde{x}\tilde{t}^{-1} \in X(A, b)$.

*(ii) If* $\tilde{u} > 0$*, then* $X(A, b) = \emptyset$.

(b) *If* $\tilde{t} = 0$*, then* $0 \neq \tilde{x} \in X(A, 0)$.

*Proof* (a) (*i*) Follows from (6a), (6b), and (6f).

(*ii*) Follows from (6c), (6e), and Lemma 3, as $(\tilde{y} - e\tilde{v}) \in Y(A, b)$.

(b) $\tilde{t} = 0 \Rightarrow$ (by (6d) and (6f)) $\tilde{y} = 0,\ \tilde{v} = 0, \Rightarrow$ (by (6e) and (6f)) $\tilde{u} = 0$. $\tilde{y} = 0,\ \tilde{u} = \tilde{v} = \tilde{t} = 0, \Rightarrow$ (by (6g)) $\tilde{x} \neq 0$. $\tilde{t} = \tilde{u} = 0 \Rightarrow$ (by (6a), (6b) and (6f)) $A\tilde{x} = 0,\ \tilde{x} \geq 0$.

□

Our main goal in this note is to show how to reduce $F(A, b)$ to a zero-sum game. Specifically, our goal is to show how one can solve any LF problem by way of solving a zero-sum game. To have a meaningful reduction, we consider, as discussed in the beginning of Sect. 3, only strongly polynomial reductions. In the rest of Sect. 3, we present two strongly polynomial reductions of LF problems to zero-sum games. The first is a *Karp type* reduction based on solving a single Dantzig game of a modified LF problem. This reduction is shown to be strongly polynomial whenever $A, b$ are composed of algebraic numbers. The second reduction is a *Cook type* reduction, where a solver for zero-sum games is used as an oracle which is called up to $n$ times, starting with the original LF problem and continues with progressively strictly smaller LF problems. This reduction is strongly polynomial even when $A, b$ are composed of real numbers.

The analogue of the Strong Duality Theorem for the FL problem is the well-known Farkas' Lemma Farkas (1902). In fact, it is well established (see e.g Dantzig (1963)) that the Strong Duality Theorem is a direct corollary of Farkas' Lemma which is presented below.

**Theorem 5** (Farkas' Lemma) $X(A, b) \neq \emptyset$ *if and only if* $Y(A, b) = \emptyset$.

In Sect. 4 we provide a proof of Farkas' Lemma that is derived directly from the Minimax Theorem. The proof is based on the observation that Ville's Theorem Ville (1938), a well-known homogenous variant of Farkas' Lemma, is a direct corollary of the Minimax Theorem. We use the following notation throughout this note. Given a matrix $A \in \mathcal{R}^{m \times n}$, and subsets $I \subseteq \{1, \ldots, m\}$ and $J \subseteq \{1, \ldots, n\}$, we denote by $A_{IJ}$ the submatrix of $A$ whose rows and columns are indexed by $I$ and $J$ respectively. When $I = \{1, \ldots, m\}$, we replace $A_{IJ}$ with $A_{\bullet J}$. We use $e$ to denote a column vector of 1's.

## 3 Reduction of LF problems to zero-sum games

In this section we discuss reductions of LF problems to zero-sum games. Typically, in complexity theory, meaningful reductions are required to be polynomial, that is, the effort of the reduction, as well as the effort to recover the solution, is required to be polynomial in the size of the problem. In addition, the size of the resultant reduction should be bounded by a polynomial function of the size of the presentation of the original problem. To make the reduction meaningful in the framework of complexity theory and given that any LF problem with rational data is known to be solved in time polynomial in the size of the problem (measured as the number of bits needed to encode the data), we consider only reductions where the effort (in terms of number of basic arithmetic operations) is bounded above by a polynomial in $m$ and $n$ (the size of the matrix of coefficients underlying the LF problem), and where the number of rows and columns of the payoff matrix of the zero-sum game(s) used in the reduction is (are) bounded above by a polynomial in $m$ and $n$. In addition, when the reduction is applied to rational input, the size of the numbers occurring during the reduction is polynomially bounded in the dimension of the input and the size of the input numbers. We call such reductions *strongly polynomial*.

In Subsect. 3.1 we present a *Karp-type* reduction in which $F(A, b)$ is reduced in time polynomial in $m$ and $n$ to a single Dantzig game whose payoff matrix is of size $(m+n+4) \times (m+n+4)$, and whose solution can be trivially converted to resolve $F(A, b)$ as specified in Proposition 4. However, this reduction is critically dependent on uniform bounding below the absolute values of the determinants of all $m \times m$ nonsingular submatrices of the underlying matrix of coefficients, which can be easily computed in strongly polynomial time for rational data. In fact, an analogue bound is applicable when the data is algebraic. However, the question of whether, for matrices whose data is real, one can find such a bound in strongly polynomial time is open. As a consequence, we are able to show that this reduction is strongly polynomial only for LF with algebraic data.

In Subsect. 3.2 we present a *Cook-type* reduction in which a zero-sum game solver is used up to $n$ times as an oracle, and where the size of the payoff matrix of each game is at most $(m+n+3) \times (m+n+3)$ and where, for rational data, the size of the matrix entries are bounded above by polynomial function of the problem's size. The reduction of $F(A, b)$ to the initial game and the conversion of the solution of the final game to a solution for $F(A, b)$ can be trivially carried out in time polynomial in $m$ and $n$. The rest of the effort of the reduction is also strongly polynomial. This reduction is applicable even when the data is composed of real numbers.

### 3.1 **Karp-type reduction**

Recall that it is shown in Proposition 4 that if the solution of the Dantzig game associated with $F(A, b)$ does not provide a resolution to $F(A, b)$, then we obtain $0 \neq \tilde{x} \in X(A, 0)$. Noting that this case occurs if and only if the set $X(A, b)$ is unbounded (whenever it is nonempty), we add the constraint $e^{\mathsf{T}}x \leq \beta$ (where $\beta > 0$) which bounds the set $X(A, b) \cap \{x \mid e^{\mathsf{T}}x \leq \beta\}$ whenever it is nonempty. To express this problem in standard form, we define

$$\hat{A} = \begin{pmatrix} A & 0 \\ e^{\mathsf{T}} & 1 \end{pmatrix}, \quad \hat{b}(\beta) = \begin{pmatrix} b \\ \beta \end{pmatrix}.$$

Now, we replace $X(A, b)$ with $X(\hat{A}, \hat{b}(\beta))$ where $\beta$ is a positive scalar.[1] In the following two propositions, we show that the Dantzig game associated with $F(\hat{A}, \hat{b}(\beta))$

[1] Note that $X(\hat{A}, \hat{b}(\beta))$ has an additional variable, $x_{n+1}$, corresponding to the last column of $\hat{A}$.

always resolves it, and that for sufficiently large $\beta$, $F(\hat{A}, \hat{b}(\beta))$ can be solved in lieu of $F(A, b)$.

**Proposition 6** *Let* $\beta > 0$. *If* $(\tilde{x}, \tilde{u}, \tilde{y}, \tilde{v}, \tilde{t}) \in G(\hat{A}, \hat{b}(\beta))$, *then* $\tilde{t} > 0$.

*Proof* Since $X(\hat{A}, \hat{b}(\beta))$ is bounded if nonempty, we have that $X(\hat{A}, 0) = \{0\}$. Thus, by Proposition 4, $\tilde{t} > 0$. □

**Proposition 7** *Let* $A \in \mathcal{R}^{m\times n}$, $b \in \mathcal{R}^m$, *where* $A_{\bullet j} \neq 0$ *for* $j = 1, \dots, n$. *Then, there exists* $\beta(A, b)$ *such that for all* $\beta \geq \beta(A, b)$, $X(A, b) \neq \emptyset$ *if and only if* $X(\hat{A}, \hat{b}(\beta)) \neq \emptyset$.

The key to the proof of Proposition 7 is a well known theorem (Carathodory (1911)) that we present as Lemma 8. For a vector $x$, we denote:

$$J^+(x) = \{j \mid x_j > 0\},\ J^-(x) = \{j \mid x_j < 0\},\ J^0(x) = \{j \mid x_j = 0\},$$
$$J^{\pm}(x) = \{j \mid x_j \neq 0\}.$$

**Lemma 8** (Caratheodory's Theorem) *Suppose* $X(A, b) \neq \emptyset$ *where* $A_{\bullet j} \neq 0$ *for* $j = 1, \dots, n$. *Then, there exists* $\bar{x} \in X(A, b)$ *such that the columns of* $A_{\bullet J^+(\bar{x})}$ *are linearly independent.*

*Proof of Proposition 7* Suppose $(\bar{x}_1 \dots \bar{x}_{n+1})^\mathsf{T} \in X(\hat{A}, \hat{b}(\beta))$, then clearly $(\bar{x}_1 \dots \bar{x}_n)^\mathsf{T} \in X(A, b)$. Next, we prove that there exists $\beta(A, b)$ such that for all $\beta \geq \beta(A, b)$, $X(A, b) \neq \emptyset$ implies $X(\hat{A}, \hat{b}(\beta)) \neq \emptyset$. Suppose $X(A, b) \neq \emptyset$ and let

$$S(A, b) = \{x \mid Ax = b, \text{ and the columns of } A_{\bullet J^{\pm}(x)} \text{ are linearly independent}\}. \tag{7}$$

By the assumption that $rank(A) = m$, we have $|S(A, b)| \geq 1$. Suppose $\tilde{x} \in S(A, b)$. Since the columns of $A_{\bullet J^{\pm}(\tilde{x})}$ are linearly independent, $\tilde{x}_{J^{\pm}(\tilde{x})}$ is the unique solution of $A_{\bullet J^{\pm}(\tilde{x})} x_{J^{\pm}(\tilde{x})} = b$. Thus, $|S(A, b)| \leq 2^n$, establishing the existence of

$$\beta(A, b) = \max_{x\in S(A,b)} e^\mathsf{T} x + 1. \tag{8}$$

Now, since $X(A, b) \neq \emptyset$ and by Lemma 8, there exists $\bar{x} \in X(A, b) \cap S(A, b)$, so $e^\mathsf{T}\bar{x} \leq \beta(A, b)$, which implies that $(\bar{x} \;\; \beta - e^\mathsf{T}\bar{x})^\mathsf{T} \in X(\hat{A}, \hat{b}(\beta))$ for all $\beta \geq \beta(A, b)$. □

In view of Propositions 4, 6, and 7, it is apparent that any solution to the Dantzig game associated with $F(\hat{A}, \hat{b}(\beta))$ with $\beta \geq \beta(A, b)$ as specified in (8), resolves $F(A, b)$. The key issue, though, is how to compute an upper bound to $\beta(A, b)$ in strongly polynomial time. Note that the members of $S(A, b)$ correspond to the basic solutions of $Ax = b$. By Cramer's rule (and by the assumption that $rank(A) = m$), we have that each component of a basic solution to $Ax = b$ is a ratio $\frac{D_2}{D_1}$ of two determinants, $D_1$ which is the determinant of an $m \times m$ nonsingular submatrix of $A$, and $D_2$ which is the same as $D_1$ with $b$ replacing one of its columns. Specifically, we can set an upper bound $\hat{\beta}$ for $\beta(A, b)$, where

$$\hat{\beta} = m\frac{\delta_{max}}{\delta_{min}} + 1$$

where

$$\delta_{max} = \max\{|det(D)| \ \text{ over all } m \times m \text{ submatrices } D \text{ of } (A \ b)\},$$
$$\delta_{min} = \min\{|det(D)| \ \text{ over all } m \times m \text{ nonsingular submatrices } D \text{ of } A\}.$$

A routinely used upper bound for $\delta_{max}$ in the LP literature is $(\sqrt{m}\theta)^m$, where $\theta$ is the largest absolute value of the entries in $(A, b)$. In addition, when the data of the problem is rational we can assume, without loss of generality, that all the entries of $(A, b)$ are integers which implies that $\delta_{min} \geq 1$. Thus, for the case of integer data, one can calculate in strongly polynomial time the following upper bound on $\hat{\beta}$ (and hence on $\beta(A, b)$):

$$\hat{\beta} \leq m(\sqrt{m}\theta)^m + 1.$$

It is shown in Adler and Beling (1997) that if the entries of $(A, b)$ are algebraic numbers, a similar upper bound to $\hat{\beta}$ can be calculated in strongly polynomial time. However, if the data is real, it is not known whether it is possible to calculate in strongly polynomial time a positive lower bound to $\delta_{min}$ and hence, an upper bound to $\hat{\beta}$ (for a discussion on those bounds see e.g. Megiddo (1990)). Thus, the reduction in this section is proved to be valid only for LP problems with algebraic data.

## 3.2 **Cook-type reduction**

In this subsection we show how to resolve $F(A, b)$ by using a zero-sum game solver as an oracle that is called at most $n$ times. Given $F(A, b)$, the reduction starts by solving the Dantzig game associated with it. By Proposition 4, we either resolve $F(A, b)$ or obtain $0 \neq \tilde{x} \in X(A, 0)$. The basis for the reduction is the observation that in the latter case, we can relax the nonnegativity of the variables indexed by $J^+(\tilde{x})$. Given $J \subseteq \{1, \ldots, n\}$, we define

$$X(A, b, J) = \{x \mid Ax = b, \ x_J \geq 0\}.$$

Specifically, we show in Proposition 9 that $X(A, b)$ can be replaced by $X(A, b, J^0(\tilde{x}))$. This new problem allows us to substitute out the variables indexed by $J^+(\tilde{x})$ from the LF problem corresponding to $X(A, b, J^0(\tilde{x}))$, leading to a strictly smaller equivalent problem. This process is repeated at most $n$ times before we necessarily resolve $F(A, b)$. We also show that the reduction is strongly polynomial and that it is valid when the data is real. We start the presentation of the reduction with the following proposition:

**Proposition 9** *Let* $0 \neq \tilde{x} \in X(A, 0)$. *Then,* $X(A, b) \neq \emptyset$ *if and only if* $X(A, b, J^0(\tilde{x})) \neq \emptyset$.

*Proof* Clearly $X(A, b) \subseteq X(A, b, J^0(\tilde{x}))$. Suppose $\hat{x} \in X(A, b, J^0(\tilde{x}))$, and that $J^-(\hat{x}) \neq \emptyset$ (so $\hat{x} \notin X(A, b)$). For $\alpha \in \mathcal{R}$, let $x(\alpha) = \hat{x} + \alpha\tilde{x}$. Thus,

$$Ax(\alpha) = A(\hat{x} + \alpha\tilde{x}) = A\hat{x} + \alpha A\tilde{x} = b + \alpha 0 = b.$$

Let $\bar{\alpha} = \max_{j \in J^-(\hat{x})}(-\frac{\hat{x}_j}{\tilde{x}_j})$. Since $\tilde{x}_{J^+}(\tilde{x}) > 0$ and $J^-(\hat{x}) \subseteq J^+(\tilde{x})$, we have $x(\alpha) = \hat{x} + \alpha\tilde{x} \geq 0$ for all $\alpha \geq \bar{\alpha}$. Thus, $x(\alpha) \in X(A, b)$ for all $\alpha \geq \bar{\alpha}$, so we conclude that $X(A, b, J^0(\tilde{x})) \neq \emptyset$ implies $X(A, b) \neq \emptyset$. □

A standard technique for handling unrestricted variables in a system of linear equations is to create an equivalent system of equations in which the unrestricted variables are substituted out of the system. Specifically, let $0 \neq \tilde{x} \in X(A, 0)$ and denote $J^+ = J^+(\tilde{x})$, $J^0 = J^0(\tilde{x})$, and $\ell = rank(A_{\bullet J^+})$ (note that $\ell \geq 1$).

If $J^0 = \emptyset$, then $F(A, b, J^0)$ corresponds to a system of linear equations with no nonnegativity constraints. Hence, it can be resolved by Gaussian elimination.

If $J^0 \neq \emptyset$, then one can compute (see e.g. Dantzig (1963)):

$$H \in \mathcal{R}^{(m-\ell)\times|J^0|},\ F \in \mathcal{R}^{m\times m},\ D \in \mathcal{R}^{|J^+|\times|J^0|},\ h \in \mathcal{R}^{(m-\ell)} \text{ and } d \in \mathcal{R}^{|J^+|}, \tag{9}$$

such that:

$$\text{If } \bar{x} \in X(A, b) \text{ then } \bar{x}_{J^0} \in X(H, h), \tag{10a}$$

$$\text{if } \bar{x}_{J^0} \in X(H, h) \text{ then, letting } \bar{x}_{J^+} = d - D\bar{x}_{J^0}, \text{ we have } \bar{x} \in X(A, b, J^0), \tag{10b}$$

$$\text{for a given } u \in \mathcal{R}^{(m-\ell)}, \text{ let } y^\mathsf{T} = (0\ \ u^\mathsf{T})F, \text{ then } y^\mathsf{T}A = (0\ \ u^\mathsf{T}H),\ y^\mathsf{T}b = u^\mathsf{T}h. \tag{10c}$$

Now, we can construct a valid reduction of $F(A, b)$ as follows:

(i) We start by solving the Dantzig's game associated with $F(A, b)$. By Proposition 4, if $F(A, b)$ is not resolved by solving the game then we obtain $0 \neq \tilde{x} \in F(A, 0)$.

(ii) Set $J^+ = J^+(\tilde{x})$, $J^0 = J^0(\tilde{x})$, and compute $H$, $D$, $F$, $h$ and $d$ as in (9).

(iii) Solve the Dantzig's game associated with $F(H, h)$.

(iv) By Proposition 4 we get one of the following three cases:

- (iv-a) We get $\bar{x}_{J^0} \in X(H, h)$. Then, by (10b) and Proposition 9, we can construct $\bar{x} \in X(A, b)$ and we are done.
- (iv-b) We get $\bar{u}$ such that $\bar{u}^\mathsf{T}H \leq 0$, $\bar{u}^\mathsf{T}h > 0$. Then, by (10c), we can construct $\bar{y}$ such that $\bar{y}^\mathsf{T}A \leq 0$ and $\bar{y}^\mathsf{T}b > 0$ (so by Lemma 3, $X(A, b) = \emptyset$) and we are done.
- (iv-c) We get $0 \neq \bar{x}_{J^0} \in X(H, 0)$. Then by (10b) and Proposition 9 we can construct $0 \neq \bar{x} \in X(A, 0)$. Note that since $\tilde{x}, \bar{x} \in X(A, 0)$ then $\tilde{x} + \bar{x} \in X(A, 0)$ and $J^+ \subset \{j \mid \tilde{x}_j + \bar{x}_j > 0\}$. We now set $\tilde{x} \leftarrow \tilde{x} + \bar{x}$ and go to (ii).

Since, by step iv-c, $|J^+|$ is monotonically increasing at each step without a resolution of $F(A, b)$, this process terminates in at most $n$ steps. The main computation of the reduction (step ii) can be carried out by Gaussian elimination within a time polynomial in $n$, where for rational data, all the numbers generated during the reduction are bounded above by a polynomial function of the problem's size. The rest of the calculations used in the reduction involve manipulating (adding, multiplying, decomposing) matrices whose number of entries is bounded above by $n^2$, and, for rational data, all numbers are bounded above by a polynomial function of the problem's size. All these manipulations can be performed within a time polynomial in $n$. If one can perform the required matrix manipulations with real numbers, then the reduction is applicable to LF problems with real data. Thus, if the game solver used in the reduction is applicable to payoff matrices composed of real numbers, a discovery of a strongly polynomial algorithm for solving zero-sum games in real numbers would imply the existence of a strongly polynomial algorithm for solving LP problems in real numbers. Note that the reduction presented in Subsect. 3.1 implies that a discovery of a strongly polynomial algorithm for solving zero-sum games could be applied only for LP problems with algebraic numbers.

Finally, we note that the reduction presented in this section provides a proof of Farkas' Lemma (Theorem 5). Specifically, according to the reduction we conclude either with $\bar{x} \in X(A, b)$ (step iv-a) or with $\bar{y}$ such that $\bar{y}^\mathsf{T} A \leq 0,\ \bar{y}^\mathsf{T} b > 0$ (step iv-b), which considering Lemma 3, completes the proof of Theorem 5. In the next section we provide a more direct proof of Farkas' Lemma.

## 4 From the minimax theorem to Farkas' lemma

As discussed in the introduction, while the Minimax Theorem easily and directly follows from Farkas' Lemma, this is not the case when going in the other direction. In this section we show that the Minimax Theorem is essentially equivalent to Villes' Theorem, a well known homogenous variant of Farkas' Lemma. We then prove Farkas' Lemma from Villes' Theorem by way of Tucker's Theorem, another well known variant of Farkas' Lemma. We start by stating the Minimax Theorem as applied to a general zero-sum game (rather than to a symmetric zero-sum game as in Theorem 1).

**Theorem 10** (Minimax Theorem—general zero-sum game)
*Given $B \in \mathcal{R}^{m\times d}$, and denoting $S(n) = \{s \in \mathcal{R}^n \mid e^\mathsf{T} s = 1,\ s \geq 0\}$,*

$$\max_{x\in S(d)} \min_{y\in S(m)} y^\mathsf{T} Bx = \min_{y\in S(m)} \max_{x\in S(d)} y^\mathsf{T} Bx \tag{11}$$

Next, we show that the well known Ville's Theorem Ville (1938) is a direct corollary of the Minimax Theorem as stated above. For $B \in \mathcal{R}^{m\times d}$ let

$$X_V(B) = \{x \mid Bx \geq 0,\ 0 \neq x \geq 0\}, \qquad Y_V(B) = \{y \mid y^\mathsf{T} B < 0,\ y \geq 0\}.$$

**Theorem 11** (Ville's Theorem) $X_V(B) \neq \emptyset$ *if and only if* $Y_V(B) = \emptyset$.

*Proof* Suppose $\bar{x} \in X_V(B)$, and $\bar{y} \in Y_V(B)$, then

$$0 \leq \bar{y}^\mathsf{T}(B\bar{x}) = (\bar{y}^\mathsf{T}B)\bar{x} < 0,$$

a contradiction. On the other hand, applying Theorem 10, we have that either the left-hand-side of (11) is greater or equal to zero, or the right-hand-side of (11) is negative. In the former $X_V(B) \neq \emptyset$, while in the latter $Y_V(B) \neq \emptyset$. □

*Remark* One can also prove the Minimax Theorem (as presented in Theorem 10) as a direct corollary of Ville's Theorem. In fact, von Neumann and Morgenstern in their seminal book on game theory von Neumann and Morgenstern (1944), use Ville's Theorem as the main step in proving the Minimax Theorem for zero-sum games. Thus, in a sense, Ville's Theorem is a "natural equivalent" theorem to the Minimax Theorem.

Given the preceding theorem, a natural next step is to prove Farkas' Lemma directly from Ville's Theorem. While many publications discuss Villes's Theorem as a by-product of Farkas' Lemma, there seems to be no discussion about deriving Farkas' Lemma directly from Ville's Theorem. This is not surprising as it is not clear how to "de-homogenize" the inequalities in $X_V(B)$. In the following we show that indeed it is possible to derive Farkas' Lemma from Ville's Theorem by way of two other classical results regarding alternative systems of linear inequalities and equations, which are known as Gordan's and Tucker's Theorems. We first introduce Gordan's Theorem Gordan (1873) which is actually Ville's Theorem formulated in standard form where $X_V(B)$ is replaced by

$$X_G(A) = \{x \mid Ax = 0,\ 0 \neq x \geq 0\}.$$

Expressing $Ax = 0$ as $Ax \geq 0,\ -e^\mathsf{T}Ax \geq 0$; setting $B = \begin{pmatrix} A \\ -e^\mathsf{T}A \end{pmatrix}$, defining $Y_G(A) = \{y \mid y^\mathsf{T}A < 0\}$, and applying Theorem 11, we get:

**Theorem 12** (Gordan's Theorem) $X_G(A) \neq \emptyset$ *if and only if* $Y_G(A) = \emptyset$.

As discussed throughout the LP literature, the following theorem, which is known as Tucker's Theorem Tucker (1956), plays a major role in proving many variants of Farkas' Lemma. Next, we show that Tucker's Theorem can be proved directly from Gordan's Theorem. For $A \in \mathcal{R}^{m\times n}$, let

$$X_T(A) = \{x \mid Ax = 0,\ x \geq 0\}, \quad Y_T(A) = \{y \mid y^\mathsf{T}A \leq 0\}.$$

**Theorem 13** (Tucker Theorem) *There exist* $\bar{x}, \bar{y}$ *such that*

$$\bar{x} \in X_T(A), \quad \bar{y} \in Y_T(A), \ \textit{and} \ \bar{x} - A^T\bar{y} > 0.$$

*Proof* Let $\{J^+, J^0\}$ be a partition of $\{1, \ldots, n\}$ where

$$J^+ = \{j \mid x_j > 0 \ \textit{for at least one}\ x \in X_T(A)\}.$$

If $J^+ = \emptyset$, then $X_T(A) = \{0\}$ which implies that $X_G(A) = \emptyset$, so by Theorem 12 there exists $\bar{y} \in Y_G(A)$. Thus $\bar{x} = 0$, together with $\bar{y}$, satisfy the theorem. Hence, we assume $J^+ \neq \emptyset$. Now, for each $j \in J^+$, let $\bar{x}^j \in X_T(A)$ be such that $\bar{x}^j_j > 0$ and let

$$\bar{x} = \sum_{j \in J^+} \bar{x}^j.$$

Since $A\bar{x} = A(\sum_{j \in J^+} \bar{x}^j) = \sum_{j \in J^+} A\bar{x}^j = 0$ and since $\bar{x} = \sum_{j \in J^+} \bar{x}^j \geq 0$, we have that $\bar{x} \in X_T(A)$ and $J^+(\bar{x}) = J^+$. If $J^0 = \emptyset$, then $\bar{x},\ \bar{y} = 0$ satisfy the theorem, so we assume $J^0 \neq \emptyset$. By the definition of $J^+$,

$$\{x \mid Ax = 0,\ x \geq 0,\ x_{J^0} \neq 0\} = \emptyset. \tag{12}$$

Using the construction in (9) and (10a–10b), we have that (12) is true if and only if

$$\{x_{J^0} \mid Hx_{J^0} = 0,\ 0 \neq x_{J^0} \geq 0\} = \emptyset. \tag{13}$$

Thus, by Theorem 12, there exist $\bar{u}$ such that $\bar{u}^\mathsf{T} H < 0$, which, by (10c), implies the existence of $\bar{y}$ such that $\bar{y}^\mathsf{T} A = (0 \quad \bar{u}^\mathsf{T} H)$, so $\bar{y} \in Y_T(A)$ with $\bar{y}^\mathsf{T} A_{\bullet J^+} = 0$ and $\bar{y}^\mathsf{T} A_{\bullet J^0} < 0$. However, since $\bar{x} \in X_T(A)$ where $\bar{x}_{J^+} > 0,\ \bar{x}_{J^0} = 0$, we have that $\bar{x} - A^\mathsf{T}\bar{y} > 0$, which completes the proof. □

Finally, we show that Farkas' Lemma can be proved directly from Theorem 13, completing the task of deriving it from the Minimax Theorem.

*Proof of Theorem 5* (Farkas' Lemma) . Consider the system of inequalities

$$(A \quad -b)\begin{pmatrix} x \\ t \end{pmatrix} = 0, \ \begin{pmatrix} x \\ t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{14}$$

and

$$y^\mathsf{T}(A \quad -b) \leq (0 \ \ 0). \tag{15}$$

By Theorem 13 there exist $\bar{x}, \bar{t}$ satisfying (14) and $\bar{y}$ satisfying (15) such that

$$\begin{pmatrix} \bar{x} \\ \bar{t} \end{pmatrix} + \begin{pmatrix} -A^\mathsf{T}\bar{y} \\ b^\mathsf{T}\bar{y} \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so

$$A\bar{x} = b\bar{t}, \quad \bar{x} \geq 0, \ \bar{t} \geq 0, \quad \bar{y}^\mathsf{T} A \leq 0, \quad \bar{y}^\mathsf{T}\bar{b} \geq 0, \quad \text{and} \ \ \bar{t} + \bar{y}^\mathsf{T} b > 0.$$

Thus, we conclude that either $\bar{t} > 0$, implying $\bar{x}\bar{t}^{-1} \in X(A, b)$, or $\bar{y}^\mathsf{T} b > 0$, implying $\bar{y} \in Y(A, b)$. Combining this with Lemma 3 complete the proof. □

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