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New characterizations of row sufficient matrices

Ilan Adler [a], Richard W. Cottle [b,*], Sushil Verma [c]

[a] *Department of Industrial Engineering and Operations Research, University of California, Berkeley, CA 94720, United States*
[b] *Department of Management Science and Engineering, Stanford University, Stanford, CA 94305-4026, United States*
[c] *Signal Demand, Inc., 340 Brannan Street, San Francisco, CA 94107, United States*

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Dedicated to the memory of a great scholar and a valued friend, Professor David Gale.

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ABSTRACT

In dealing with a linear complementarity problem, much depends on knowing that the matrix, through which the particular LCP is defined, belongs to a suitable matrix class. Two such classes are **SU** – the so-called sufficient matrices – and **L** which were introduced in [R.W. Cottle, J.-S. Pang, V. Venkateswaran, Sufficient matrices and the linear complementarity problem, Linear Algebra Appl. 114/115 (1989) 231–249; B.C. Eaves, The linear complementarity problem, Manage. Sci. 17 (1971) 612–634], respectively. In an earlier article [I. Adler, R.W. Cottle, S. Verma, Sufficient matrices belong to **L**, Math. Prog. 106 (2006) 391–401], the authors proved that **SU** is a subclass of **L**. By definition, the class **SU** is the intersection of two distinct classes: **RSU**, the row sufficient matrices, and **CSU**, the column sufficient matrices. In the present work, we strengthen the aforementioned inclusion by showing that all row sufficient matrices belong to **L**. Using what we call "structural properties" of certain matrix classes, we add to the existing characterizations of **RSU** in [R.W. Cottle, S.-M. Guu, Two characterizations of sufficient matrices, Linear Algebra Appl. 170 (1992) 65–74; S.-M. Guu, R.W. Cottle, On a subclass of $\mathbf{P_0}$, Linear Algebra Appl. 223/224 (1995) 325–335; H. Väliaho, Criteria for sufficient matrices, Linear Algebra Appl. 233 (1996) 109–129]. This line of inquiry was inspired by asking: what must be true of a row sufficient **L**-matrix? We establish three new characterizations of **RSU** in terms of the matrix classes **L**, $\mathbf{E_0}$, and $\mathbf{Q_0}$ and the structural properties of sign-change invariance, completeness, and fullness. The new characterizations of **RSU** provide new characterizations of **SU** by adjoining a fourth structural property we call reflectiveness.

* Corresponding author. Tel.: +1 650 725 0558; fax: +1 650 723 2826.
E-mail addresses: adler@newton.berkeley.edu (I. Alder), rwc@stanford.edu (R.W. Cottle), Sushil.Verma@SignalDemand.com (S. Verma).

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1. Introduction

Matrix classes play a prominent role in the theory of the *Linear Complementarity Problem* (LCP). At the center of the present investigation is the class **RSU** of row sufficient matrices which was introduced in [9] and later characterized in [7,13,19]. The importance of this class is underscored by its connection with the existence of solutions as well as the applicablity of Lemke's Method [15] and the Principal Pivoting Method [6,3] for constructively solving instances of such LCPs. Sharing the spotlight with **RSU** is the class **L** introduced by Eaves [10]. The present group of authors explored these two classes in [1]. There, we showed that if a matrix M and its transpose belong to **RSU**, then it must belong to **L**. In the present paper we strengthen that theorem by establishing the (proper) inclusion of the entire class **RSU** in **L**. This raises the question: Given that a matrix M belongs to **L**, what more must M satisfy to belong to **RSU**? Our responses to this and other such questions employ what we call "structural properties" of certain matrix classes. Four of these structural properties play important parts in the development.

Our answer to the question posed above is that **RSU** is precisely the class of fully-completely-**L** matrices. While researching and demonstrating this characterization, it became natural to ask similar questions about other classes that contain **RSU**. This investigation led to the identification of new matrix classes, structural properties, matrix class inclusions, and further characterizations of **RSU**.

2. Notation and terminology

In this section we assemble the definitions and notations needed for reading the paper. Most (but not all) of these can be found in [8].

The *Linear complementarity problem* can be stated as follows: given $M \in \mathcal{R}^{n\times n}$ and $q \in \mathcal{R}^{n}$, find a vector $z \in \mathcal{R}^{n}$ such that

$$z \geqslant 0, \tag{1}$$

$$q + Mz \geqslant 0, \tag{2}$$

$$z^{\mathrm{T}}(q + Mz) = 0. \tag{3}$$

We denote this system by the pair (q, M). A vector z satisfying (1) and (2) is said to be *feasible*, and the set of all feasible vectors for the LCP (q, M) is denoted FEA(q, M). The solution set of (q, M) is denoted SOL(q, M). LCPs of the form $(0, M)$ are called *homogeneous*. Because they are of special interest, we denote the set of *nonzero* $z \in$ SOL$(0, M)$ by SOL$_+(0, M)$.

For any nonzero vector $z \in \mathcal{R}^{n}$, the (nonempty) index set $\sigma(z) = \{i : i \in \{1, 2, \ldots, n\},\ z_i \neq 0\}$ is called the *support* of z.

2.1. Principal transformations

An equivalent formulation of (q, M) is the system

$$w = q + Mz, \tag{4}$$

$$w,\ z \geqslant 0, \tag{5}$$

$$z^{\mathrm{T}}w = 0. \tag{6}$$

For $M \in \mathcal{R}^{n\times n}$ and every $\alpha \subseteq \{1, 2, \ldots, n\}$ there is a corresponding principal submatrix of M denoted $M_{\alpha\alpha}$ formed by taking the elements m_{ij} of M that come from the rows $i \in \alpha$ and columns $j \in \alpha$. The determinant of a principal submatrix is called a *principal minor* of M. When the principal submatrix is nonsingular (principal minor is nonzero), there is a corresponding *principal pivotal transformation* of the system given by

$$\begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\beta} \\ M_{\beta\alpha} & M_{\beta\beta} \end{bmatrix} \xrightarrow{\wp_\alpha} \begin{bmatrix} M_{\alpha\alpha}^{-1} & -M_{\alpha\alpha}^{-1}M_{\alpha\beta} \\ M_{\beta\alpha}M_{\alpha\alpha}^{-1} & M_{\beta\beta} - M_{\beta\alpha}M_{\alpha\alpha}^{-1}M_{\alpha\beta} \end{bmatrix} \tag{7}$$

and

$$\begin{bmatrix} q_\alpha \\ q_\beta \end{bmatrix} \xrightarrow{\wp_\alpha} \begin{bmatrix} -M_{\alpha\alpha}^{-1} q_\alpha \\ q_\beta - M_{\beta\alpha} M_{\alpha\alpha}^{-1} q_\alpha \end{bmatrix}, \tag{8}$$

where β denotes the *complement* of the index set α.

Such an operation is called *principal pivoting*; the matrix $M_{\alpha\alpha}$ is called the *pivot block*. The operator $\wp_\alpha$ acts on the data in the system (4). Accordingly, the LCP (q, M) goes into the LCP $\wp_\alpha(q, M) = (\bar{q}, \overline{M})$ where $\bar{q}$ is given by the right-hand side of (8) and $\overline{M}$ is given by the right-hand side of (7). It is convenient to allow the abuse of language $\overline{M} = \wp_\alpha(M)$ and $\bar{q} = \wp_\alpha(q)$.

The system (4) can be expressed in slightly greater detail as

$$w_\alpha = q_\alpha + M_{\alpha\alpha} z_\alpha + M_{\alpha\beta} z_\beta, \tag{9}$$

$$w_\beta = q_\beta + M_{\beta\alpha} z_\alpha + M_{\beta\beta} z_\beta. \tag{10}$$

We may think of $\wp_\alpha(q, M)$ as the data we would obtain by solving the system (4) for the subvector z_α in terms of w_α and z_β and then substituting the latter expression for z_α in (10).

Another commonly used operation is called *principal rearrangement*. This involves permutation of the rows and columns of the data of an LCP. Thus, if P is a permutation matrix, the corresponding principal rearrangement sends (q, M) into (Pq, PMP^{T}). These two LCPs are equivalent with respect to feasibility and solvability. Using a suitable permutation P, we can regard any principal submatrix of M as the corresponding *leading* principal submatrix of PMP^{T}.

2.2. Classes of matrices

The following are criteria for an $n \times n$ matrix M to belong to one of the subclasses of $\mathcal{R}^{n\times n}$ appearing in this study. For a comprehensive list of matrix classes in the LCP, see [5].

$M \in \mathbf{P_0}$	iff all its principal minors are nonnegative.
$M \in \mathbf{PSD}$	iff $z^T Mz \geqslant 0$ for all z.
$M \in \mathbf{CSU}$	(is *column sufficient*) iff $z_i(Mz)_i \leqslant 0$ for all $i = 1, 2, \ldots, n$ implies that $z_i(Mz)_i = 0$ for all $i = 1, 2, \ldots, n$.
$M \in \mathbf{RSU}$	(is *row sufficient*) iff $M^{\mathrm{T}} \in \mathbf{CSU}$.
$M \in \mathbf{SU}$	(is *sufficient*) iff $M \in \mathbf{RSU} \cap \mathbf{CSU}$.
$M \in \mathbf{E_0}$	(is *semimonotone*) iff for every $0 \neq z \geqslant 0$ there exists some i such that $z_i > 0$ and $(Mz)_i \geqslant 0$.
$M \in \mathbf{E_1}$	iff for every vector $z \in \mathrm{SOL}_+(0, M)$, there exists non-negative diagonal matrices Λ and Ω such that $\Omega z \neq 0$ and $(\Lambda M + M^{\mathrm{T}}\Omega)z = 0$.
$M \in \mathbf{L}$	iff $M \in \mathbf{E_0} \cap \mathbf{E_1}$.
$M \in \mathbf{Q_0}$	iff $\mathrm{FEA}(q, M) \neq \emptyset$ implies $\mathrm{SOL}(q, M) \neq \emptyset$.
$M \in \mathbf{Q_0^+}$	iff $M \in \mathbf{Q_0}$ and all the diagonal elements of M are nonnegative.
$M \in \mathbf{T}$	(*has property* (T)) iff for every nonempty subset $\alpha \subset \{1, 2, \ldots, n\}$ the existence of a solution to the system $M_{\alpha\alpha} z_\alpha \leqslant 0, \quad M_{\beta\alpha} z_\alpha \geqslant 0, \quad z_\alpha > 0$ implies the existence of a vector y_α such that $(M_{\alpha\alpha})^{\mathrm{T}} y_\alpha = 0, \quad (M_{\alpha\beta})^{\mathrm{T}} y_\alpha \leqslant 0,\ 0 \neq y_\alpha \geqslant 0$.

A few remarks about these classes are in order. Regarding the class $\mathbf{SU}$, it is known [20] that $\mathbf{SU} = \mathbf{P}_*$, the latter being a matrix class introduced in [14] and having an entirely different sort of definition. Nonetheless, it was shown in [1] that $\mathbf{SU} \subset \mathbf{L}$. This inclusion did much to stimulate the questions studied in the present paper. It is a simple consequence of the definition of $\mathbf{Q_0}$ that a matrix M belongs to $\mathbf{Q_0}$ if and only if the union of the complementary cones (see [17,8, p. 17]) corresponding to M is convex. (We invoke this fact in one of our results.) The class $\mathbf{Q_0^+}$ is a new specialization of a familiar matrix class. The class $\mathbf{T}$ is just a formalization of the class of matrices having an equivalent version of property (T) which first appeared in [2].

2.3. Structural properties of classes of matrices

Section 1 alluded to four structural properties of some matrix classes that play a key role in our results. We define them now using the symbol $\mathbf{Y}$ to denote a generic class of square matrices.

Sign-change invariance. A matrix M belonging to a class $\mathbf{Y}$ is said to be *sign-change invariant*-$\mathbf{Y}$ if the matrix $SMS \in \mathbf{Y}$ for every diagonal matrix S such that $S^2 = I$, the identity matrix. The class of all sign-change invariant-$\mathbf{Y}$ matrices is denoted $\mathbf{Y}^{\mathbf{s}}$. To say that $\mathbf{Y}$ is a *sign-change invariant class* means that $\mathbf{Y} = \mathbf{Y}^{\mathbf{s}}$.

Completeness. A matrix M belonging to a class $\mathbf{Y}$ is said to be *completely*-$\mathbf{Y}$ if every principal submatrix of M also belongs to $\mathbf{Y}$. The class of all completely-$\mathbf{Y}$ matrices is denoted $\mathbf{Y}^{\mathbf{c}}$. To say that $\mathbf{Y}$ is a *complete class* means that $\mathbf{Y} = \mathbf{Y}^{\mathbf{c}}$.

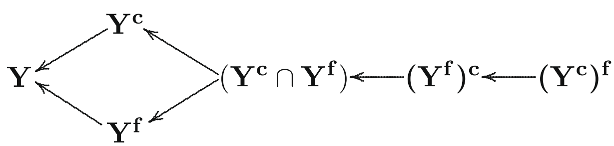
Fullness. A matrix M belonging to a class $\mathbf{Y}$ is said to be *fully*-$\mathbf{Y}$ if for every nonsingular principal submatrix of M the associated principal pivotal transform of M also belongs to $\mathbf{Y}$. The class of all fully-$\mathbf{Y}$ matrices is denoted $\mathbf{Y}^{\mathbf{f}}$. To say that $\mathbf{Y}$ is a *full class* means that $\mathbf{Y} = \mathbf{Y}^{\mathbf{f}}$.

Reflectiveness. A matrix M belonging to a class $\mathbf{Y}$ is said to be *reflectively*-$\mathbf{Y}$ if $M^{\mathrm{T}} \in \mathbf{Y}$. The class of all reflectively-$\mathbf{Y}$ matrices is denoted $\mathbf{Y}^{\mathbf{r}}$. To say that $\mathbf{Y}$ is a *reflective class* means that $\mathbf{Y} = \mathbf{Y}^{\mathbf{r}}$.

Remark 2.1. The matrix class $\mathbf{P_0}$ possesses all four of these structural properties, but this cannot be said of all the classes in our list above. An important case in point is the class **RSU** which, in addition to being a subclass of $\mathbf{P_0} \cap \mathbf{Q_0}$, possesses all but the fourth property, reflectiveness. However, the class **SU** of sufficient matrices is just $\mathbf{RSU}^{\mathbf{r}}$.

Remark 2.2. The properties of completeness and fullness for matrix classes have a solid place in the literature of linear complementarity. The symbol $\mathbf{Y}^{\mathbf{c}}$ used here to indicate the completely-$\mathbf{Y}$ matrix class is a departure from the traditional notation $\overline{\mathbf{Y}}$. The new notation gives greater stylistic consistency to our presentation.

Remark 2.3. For a matrix class $\mathbf{Y}$, the notation $\mathbf{Y}^{\mathbf{cf}}$ is read "fully-completely-$\mathbf{Y}$," meaning $(\mathbf{Y}^{\mathbf{c}})^{\mathbf{f}}$, the class of all completely-$\mathbf{Y}$ matrices that are invariant under principal pivoting. In general, the application of more than one such superscript should be interpreted from left to right. It is not always the case that the superscripts "commute", but, thanks to R.E. Stone [18], for any matrix class $\mathbf{Y}$, we can demonstrate the validity of the inclusions shown in the following diagram.



3. Preliminary results on classes of matrices

The following is apparently a well known result for which we failed to find a clear reference.

Proposition 3.1. *The class* $\mathbf{E_0}$ *is complete*.

Proof. Let $M \in \mathcal{R}^{n\times n} \cap \mathbf{E_0}$. We have to show that every proper principal submatrix of M belongs to $\mathbf{E_0}$. It is clear that all the diagonal elements of M are nonnegative and that regarded as 1×1 matrices, they belong to $\mathbf{E_0}$. We now consider an arbitrary $p \times p$ principal submatrix $M_{\alpha\alpha}$ of M where $1 < p < n$. We may assume without loss of generality that $\alpha = \{1, \ldots, p\}$. Now take an arbitrary nonzero nonnegative p-vector z_α. Extending z_α to the nonzero nonnegative n-vector $z = (z_\alpha, 0)$, we see that there exists an index i such that $z_i > 0$ and $(Mz)_i \geqslant 0$. Since i must belong to α, it follows that $M_{\alpha\alpha} \in \mathbf{E_0}$. Hence $\mathbf{E_0}$ is complete. $\square$

Our next objective is to prove that the class $\mathbf{E_1}$ is full. This requires showing that every principal pivotal transform of an $\mathbf{E_1}$-matrix is another $\mathbf{E_1}$-matrix.

Consider the equation

$$w = Mz \tag{11}$$

and let $M_{\alpha\alpha}$ be a nonsingular principal submatrix of M. Then it is possible to execute a principal pivot transformation $\wp_\alpha$ using $M_{\alpha\alpha}$ as the pivot block. It is convenient, but not restrictive, to assume that $M_{\alpha\alpha}$ is a leading principal submatrix of M. Then, letting $\overline{M} = \wp_\alpha(M)$, we can write the transformed system as

$$\bar{w} = \overline{M}\bar{z}, \tag{12}$$

where

$$\bar{w} = (z_\alpha, w_\beta) \quad \text{and} \quad \bar{z} = (w_\alpha, z_\beta). \tag{13}$$

Remark 3.2. Throughout this paper we regard all vectors as columns. The representation such as that of $\bar{w}$ and $\bar{z}$ in (13) is meant to avoid transposes and save vertical space.

Next we state an alternative characterization of the class $\mathbf{E_1}$ which, in essence, was made long ago by Garcia [12].

Proposition 3.3. *If $M \in \mathcal{R}^{n\times n}$, then $M \in \mathbf{E_1}$ if and only if for every $z \in \mathrm{SOL}_+(0, M)$, there exists a vector y such that*:

(a) $M^{\mathrm{T}} y \leqslant 0, \;\; 0 \neq y \geqslant 0$;
(b) $\sigma(y) \subseteq \sigma(z)$;
(c) $\sigma(M^{\mathrm{T}} y) \subseteq \sigma(Mz)$.

Before coming to the previously announced result on the fullness of $\mathbf{E_1}$, we recall and extend a few notions from the literature. In [1, p. 394], we defined – for any $M \in \mathcal{R}^{n\times n}$ – the polyhedral cone

$$T(M) = \mathrm{FEA}(0, -M^{\mathrm{T}}).$$

We observed that

$$\mathrm{SOL}(0, -M^{\mathrm{T}}) \subseteq \mathrm{FEA}(0, -M^{\mathrm{T}}) = T(M).$$

To capture the *nonzero* elements of $T(M)$, we now write $T_+(M)$. The vector y in condition (a) of the above proposition is such an element.

We pause a little longer to point out that when $M_{\alpha\alpha}$ is a nonsingular principal submatrix of an arbitrary square matrix M (not necessarily in $\mathbf{E_1}$), it is not generally true that $(\wp_\alpha(M))^{\mathrm{T}}$ and $\wp_\alpha(M^{\mathrm{T}})$ are the same matrix. This can be seen by considering the case of a nondiagonal 2×2 matrix. As found in [3, Theorem 1], the correct relationship is given by the formula

$$(\wp_\alpha(M))^{\mathrm{T}} = S_\beta(\wp_\alpha(M^{\mathrm{T}}))S_\beta \tag{14}$$

where S_β denotes the diagonal sign-changing matrix with entries

$$s_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \in \alpha \\ -1 & \text{if } i = j \in \beta \end{cases}$$

We are now in a position to state and prove

Proposition 3.4. *The class $\mathbf{E_1}$ is full.*

Proof. Let $M_{\alpha\alpha}$ be a nonsingular principal submatrix of the $n \times n$ matrix $M \in \mathbf{E_1}$, and let $\overline{M} = \wp_\alpha(M)$. We have to show that if $\bar{z} \in \mathrm{SOL}_+(0, \overline{M})$, then there exists a vector $\bar{y} \in T_+(\overline{M})$ such that

$$\sigma(\bar{y}) \subseteq \sigma(\bar{z}), \tag{15}$$
$$\sigma\left(\overline{M}^{\mathrm{T}}\bar{y}\right) \subseteq \sigma(\overline{M}\bar{z}). \tag{16}$$

We define $\bar{w} = \overline{M}\bar{z}$. By pivoting on $\overline{M}_{\alpha\alpha}$ we obtain a nonzero solution z of $(0, M)$. In more detail, we have

$$w = Mz, \quad w \geqslant 0,\ z \geqslant 0, \quad z^{\mathrm{T}}w = 0,$$

with $w = (w_\alpha, w_\beta) = (\bar{z}_\alpha, \bar{w}_\beta)$ and $z = (z_\alpha, z_\beta) = (\bar{w}_\alpha, \bar{z}_\beta)$. Furthermore, since $M \in \mathbf{E_1}$, there exists a vector $y \in T_+(M)$ such that

$$\sigma(y) \subseteq \sigma(z), \tag{17}$$
$$\sigma\left(M^T y\right) \subseteq \sigma(Mz). \tag{18}$$

By pivoting on $(M^{\mathrm{T}})_{\alpha\alpha}$ in the system

$$x = M^{\mathrm{T}}y, \quad x \leqslant 0,\ y \geqslant 0 \tag{19}$$

we obtain

$$\tilde{x} = \overline{(M^{\mathrm{T}})}\tilde{y} = S_\beta\left(\overline{M}^{\mathrm{T}}\right)S_\beta\tilde{y}.$$

Because $(S_\beta)^2 = I$, we have

$$S_\beta\tilde{x} = \left(\overline{M}^{\mathrm{T}}\right)S_\beta\tilde{y}.$$

From the principal pivot done in (19), we have

$$\tilde{x} = (y_\alpha, x_\beta), \quad \tilde{y} = (x_\alpha, y_\beta).$$

Thus,

$$S_\beta\tilde{x} = (y_\alpha, -x_\beta), \quad S_\beta\tilde{y} = (x_\alpha, -y_\beta).$$

Now, defining

$$\bar{x} = -S_\beta\tilde{x} = (-y_\alpha, x_\beta),$$
$$\bar{y} = -S_\beta\tilde{y} = (-x_\alpha, y_\beta),$$

we obtain a vector $\bar{y} \in T_+(\overline{M}^{\mathrm{T}})$. That is, $\bar{x} = \overline{M}^{\mathrm{T}}\bar{y}$, $\bar{x} \leqslant 0$, $\bar{y} \geqslant 0$ and $\bar{y} \neq 0$ since $y \neq 0$. Moreover, by the sequence of definitions and the inclusions (17), (18), it follows that the required inclusions (15) and (16) are satisfied. Hence $\overline{M} \in \mathbf{E_1}$. □

In the next section we will apply the following proposition, characterizing $\mathbf{P_0} \cap \mathbf{Q_0}$, which succinctly paraphrases the main result of [2, see p. 230].

Proposition 3.5. $\mathbf{P_0} \cap \mathbf{Q_0} = \mathbf{P_0} \cap \mathbf{T^f}$.

4. New characterizations of RSU

Our objective in this section is to provide three new characterizations of **RSU**. To show that M belongs to **RSU**, it suffices to prove that every one of its principal pivotal transforms is "**RSU** of order 2." It will be helpful to make the terminology more precise by recalling the

Definition. Let **Y** be a class of square matrices of all orders $n \geqslant 1$. An $n \times n$ matrix M is said to be **Y** *of order* r, $1 \leqslant r \leqslant n$, if every $r \times r$ principal submatrix of M belongs to **Y**. When $r = n$, this statement refers to the matrix M itself. We denote the class of all matrices that are **Y** of order r by $\mathbf{Y_{[r]}}$.

In the following, we shall invoke the following characterization theorems of Cottle and Guu [7].

Theorem 4.1. *The matrix $M \in \mathcal{R}^{n\times n}$ is row sufficient if and only if every one of its principal pivotal transforms is row sufficient of order* 2.

Theorem 4.2. *The matrix $N \in \mathcal{R}^{2\times 2}$ is row sufficient if and only if for every principal pivotal transform $\overline{N}$ of N :*

(i) $\overline{N} \in \mathbf{P_0}$;
(ii) *no principal rearrangement of $\overline{N}$ has the form*

$$\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \quad \text{where } b \neq 0 \leqslant a.$$

The class **RSU** is well known to contain some important matrix classes, See [4, p. 246]. Now, as a method for obtaining new characterizations of **RSU**, we identify several other subclasses of row sufficient matrices.

4.1. First characterization: $\mathbf{RSU} = \mathbf{L^{cf}}$

Theorem 4.3. $\mathbf{P_0} \cap \mathbf{E_1^{cf}} \subset \mathbf{RSU}$.

Proof. Let $M \in \mathbf{P_0} \cap \mathbf{E_1^{cf}}$. If $n = 1$, then M must be **PSD** and hence in **RSU**. Suppose $n \geqslant 2$, then since $\mathbf{P_0} \cap \mathbf{E_1^{cf}} = (\mathbf{P_0} \cap \mathbf{E_1})^{\mathbf{cf}}$, we have that the principal pivoting transformation $\overline{M}$ of any 2×2 principle submatrix N of a principal transformation of M is in $\mathbf{P_0} \cap \mathbf{E_1} \subset \mathbf{E_0} \cap \mathrm{E}_1 = \mathbf{L} \subset \mathbf{Q_0} \cap \mathrm{E}_1$. Thus, in view of Theorems 4.1 and 4.2, all we need to show is that the forbidden sign pattern (Theorem 4.2(ii)) cannot arise in $\mathbf{Q_0} \cap \mathbf{E_1}$. Appropriately, this can be seen from the following two facts: If $b > 0$, the matrix $\overline{M} \notin \mathbf{Q_0}$ as the union of the complementary cones corresponding to $\overline{M}$ is nonconvex. If $b < 0$, the matrix $\overline{M} \notin \mathbf{E_1}$, for if $z \in \mathrm{SOL}_+(0, \overline{M})$, then $z \simeq (0, +)$[1]. Let Λ and Ω be nonnegative diagonal matrices of order 2 which together with z satisfy the conditions guaranteed by the membership of $\overline{M}$ in $\mathbf{E_1}$. Then we have $\Lambda\overline{M}z = 0$ and $\Omega z = (0, \omega_2) \simeq (0, +)$. Thus,

$$0 = (\Lambda\overline{M} + \overline{M}^{\mathrm{T}}\Omega)z = \overline{M}^{\mathrm{T}}\Omega z \simeq (-, 0) \neq 0,$$

a contradiction. □

It is shown in [16] that $\mathcal{R}^{2\times 2} \cap \mathbf{E_0^f} = \mathcal{R}^{2\times 2} \cap \mathbf{P_0}$. Thus, since $\mathbf{E_0^f} \cap \mathbf{E_1^{cf}} \subseteq \mathbf{Q_0}$, we can use the proof of the preceding theorem to deduce

Corollary 4.1. $\mathbf{E_0^f} \cap \mathbf{E_1^{cf}} \subseteq \mathbf{RSU}$.

Noting that

$$\mathbf{E_0^f} \cap \mathbf{E_1^{cf}} = \mathbf{E_0^{cf}} \cap \mathbf{E_1^{cf}} = (\mathbf{E_0} \cap \mathbf{E_1})^{\mathbf{cf}} = \mathbf{L^{cf}},$$

we obtain

Corollary 4.2. $\mathbf{L^{cf}} \subseteq \mathbf{RSU}$.

Next, we establish the reverse inclusion, namely that $\mathbf{RSU} \subseteq \mathbf{L^{cf}}$.

Since the class **RSU** is both complete and full and in view of Corollary 4.1 it will be sufficient to show that $\mathbf{RSU} \subset \mathbf{L}$ (strengthening our previous result [1] that $\mathbf{SU} \subset \mathbf{L}$). The key to this is the following lemma.

[1] We use the symbol "$\simeq$" to mean "has the sign pattern".

Lemma 4.1. $\mathbf{T^s} \subset \mathbf{E_1}$.

Proof. If $\mathrm{SOL}_+(0, M) = \emptyset$, there is nothing to prove. Assume $z \in \mathrm{SOL}_+(0, M)$. Let $\alpha = \sigma(z)$ and let $\{\gamma, \delta\}$ be a partition of β (the complement of α) such that $M_{\gamma\alpha}z_\alpha = 0$ and $M_{\delta\alpha}z_\alpha > 0$. Let $\nu \subseteq \gamma$, and let D_ν denote the $n \times n$ sign-changing matrix whose negative diagonal elements are in the rows indexed by ν. Since $z \in \mathrm{SOL}_+(0, D_\nu M D_\nu)$ and $M \in \mathbf{T^s}$, there exists a vector y_α^ν satisfying the system

$$((D_\nu M D_\nu)_{\alpha\alpha})^{\mathrm{T}} y_\alpha^\nu = 0, \quad ((D_\nu M D_\nu)_{\alpha\gamma})^{\mathrm{T}} y_\alpha^\nu = 0,$$
$$((D_\nu M D_\nu)_{\gamma\delta})^{\mathrm{T}} y_\alpha^\nu \leqslant 0, \quad 0 \neq y_\alpha^\nu \geqslant 0.$$

When the definition of ν and the associated matrix D_ν are taken into account, the above system can be written as

$$(M_{\alpha\alpha})^{\mathrm{T}} y_\alpha^\nu = 0, \tag{20}$$
$$(M_{\alpha\delta})^{\mathrm{T}} y_\alpha^\nu \leqslant 0, \tag{21}$$
$$S_\nu (M_{\alpha\gamma})^{\mathrm{T}} y_\alpha^\nu \leqslant 0, \tag{22}$$
$$0 \neq y_\alpha^\nu \geqslant 0 \tag{23}$$

(where S_ν denotes the diagonal submatrix of D_ν corresponding to the index set α).

Let $\mathscr{G}$ denote the set of all nonempty subsets of γ. Given a set of solutions y_α^ν to system (20)–(23) for all $\nu \in \mathscr{G}$ (as guaranteed by the membership of M in $\mathbf{T^s}$), we claim that there exist scalars $\lambda_\nu (\nu \in \mathscr{G})$ that solve the system

$$\sum_{\nu \in \mathscr{G}} \lambda_\nu = 1, \quad \lambda_\nu \geqslant 0 \text{ for all } \nu \in \mathscr{G}, \quad (M_{\alpha\gamma})^{\mathrm{T}} \sum_{\nu \in \mathscr{G}} y_\alpha^\nu \lambda_\nu = 0. \tag{24}$$

Suppose to the contrary that system (24) has no solution. Then the corresponding homogeneous system has no nonzero solution. Accordingly, it follows from Gordan's theorem of the alternative (see [8, Section 2.7.10]) that there exists a vector u such that

$$u^{\mathrm{T}} (M_{\alpha\gamma})^{\mathrm{T}} y_\alpha^\nu < 0 \quad \text{for all } \nu \in \mathscr{G}. \tag{25}$$

Now, let μ be the set of indices $i \in \gamma$ for which $u_i > 0$; then $u^{\mathrm{T}} S_\mu \leqslant 0$. In light of (22) this leads to

$$0 \leqslant u^{\mathrm{T}} S_\mu S_\mu (M_{\alpha\gamma})^{\mathrm{T}} y_\alpha^\mu = u^{\mathrm{T}} (M_{\alpha\gamma})^{\mathrm{T}} y_\alpha^\mu,$$

contradicting (25). Thus, letting $y_\alpha = \sum_{\nu \in \mathscr{G}} y_\alpha^\nu \lambda_\nu$ and exploiting (20) and (21) we have

$$(M_{\alpha\alpha})^{\mathrm{T}} y_\alpha = 0, \quad (M_{\alpha\gamma})^{\mathrm{T}} y_\alpha = 0, \quad (M_{\alpha\delta})^{\mathrm{T}} y_\alpha \leqslant 0, \quad 0 \neq y_\alpha \geqslant 0.$$

Finally, setting $y = (y_\alpha, 0)$ completes the proof. □

Lemma 4.1 leads to the aforementioned strengthening of the inclusion $\mathbf{SU} \subset \mathbf{L}$.

Theorem 4.4. $\mathbf{RSU} \subset \mathbf{L}$.

Proof. It is well known (see [8]) that $\mathbf{RSU} \subset \mathbf{P_0} \cap \mathbf{Q_0}$; thus, in view of Proposition 3.5 and the fact that $\mathbf{P_0} \cap \mathbf{T^f} \subset \mathbf{P_0} \cap \mathbf{T}$, we know that $\mathbf{RSU} \subset \mathbf{T}$. From Lemma 4.1 and the easily verified fact that $\mathbf{RSU}$ is sign-change invariant, we have $\mathbf{RSU} \subset \mathbf{E_1}$. Finally, the fact that $\mathbf{RSU} \subset \mathbf{P_0} \subset \mathbf{E_0}$ establishes that

$$\mathbf{RSU} \subset \mathbf{E_0} \cap \mathbf{E_1} = \mathbf{L}. \qquad \square$$

Now, we are in a position to prove our first characterization of the class **RSU**

Theorem 4.5. $\mathbf{RSU} = \mathbf{L^{cf}}$.

Proof. Noting that the class **RSU** is both complete and full and considering Theorem 4.4, we have $\mathbf{RSU} \subseteq \mathbf{L^{cf}}$. Corollary 4.2 completes the proof. □

4.2. Second characterization: $\mathbf{RSU} = (\mathbf{E_0} \cap \mathbf{Q_0})^{\mathbf{s}}$

Theorem 4.6. $(\mathbf{P_0} \cap \mathbf{Q_0})^{\mathbf{s}} \subseteq \mathbf{RSU}$.

Proof. Let $M \in \mathcal{R}^{n\times n} \cap (\mathbf{P_0} \cap \mathbf{Q_0})^{\mathbf{s}}$. If $n = 1$, then M must be **PSD** and hence in **RSU**. Suppose $n \geqslant 2$; then $\mathbf{P_0} \cap \mathbf{Q_0^s} = (\mathbf{P_0} \cap \mathbf{Q_0})^{\mathbf{s}}$. Suppose that $M \notin \mathbf{RSU}$, then by Theorems 4.1 and 4.2 there exist a principal pivot transformation (possibly principally rearranged) $\overline{M}$ of M and a $\delta \subset \{1, 2, \ldots, n\}$, with $|\delta| = 2$, such that

$$\overline{M}_{\delta\delta} = \begin{bmatrix} 0 & b \\ 0 & a \end{bmatrix} \quad b \neq 0 \leqslant a.$$

In fact, it is not restrictive to assume $\delta = \{1, 2\}$.

In the following we shall use the characterization of $\mathbf{P_0} \cap \mathbf{Q_0}$ as introduced by Aganagic and Cottle [2] to show that M can not belong to $\mathbf{P_0} \cap \mathbf{Q_0}$. Let $\gamma = \{3, 4, \ldots, n\}$, and let $\rho = \{2, 3, \ldots, n\}$. Since $\overline{M}$ belongs to the full class $(\mathbf{P_0} \cap \mathbf{Q_0})^{\mathbf{s}}$, we can assume, without loss of generality, that $\overline{M}_{12} = b > 0$ and that $\overline{M}_{\gamma 1} \geqslant 0$. (Pre- and post-multiplication by a suitable sign-changing matrix will make the assumed inequalities hold.) Now, consider $z_1 = 1$. Then we have, $\bar{m}_{11} z_1 = 0$ and $\overline{M}_{\rho 1} z_1 \geqslant 0$. Thus, since $M \in \mathbf{P_0} \cap \mathbf{Q_0}$ and by [2], there should exist $y_1 > 0$ such that $y_1 \bar{m}_{11} = 0$ and $y_1 [\bar{m}_{11}\ \bar{m}_{12}\ \cdots \bar{m}_{1n}] \leqslant 0$. However, the preceding inequality is impossible since $\bar{m}_{12} = b > 0$. Hence $M \in \mathbf{RSU}$. □

Using a characterization of $\mathbf{P_0}$ due to Fiedler and Pták [11] (specifically [8, Theorem **3.4.2 (b)**]), it is easy to prove that $\mathbf{E_0^s} = \mathbf{P_0}$. Hence Theorem 4.6 leads to the

Corollary 4.3. $(\mathbf{E_0} \cap \mathrm{Q}_0)^{\mathbf{s}} \subseteq \mathbf{RSU}$.

Theorem 4.7. $\mathbf{RSU} = (\mathbf{E_0} \cap \mathbf{Q}_0)^{\mathbf{s}}$.

Proof. Considering the definition of the class **L**, the well known result (see [10]) that $\mathbf{L} \subset \mathbf{Q_0}$, and Theorem 4.4, we have

$$\mathbf{RSU} \subseteq \mathbf{L} = \mathbf{E_0} \cap \mathbf{E_1} \subseteq \mathbf{E_0} \cap \mathbf{Q_0}.$$

Noticing that the class **RSU** is sign-change invariant and considering Corollary 4.3, we conclude that $\mathbf{RSU} = (\mathbf{E_0} \cap \mathbf{Q_0})^{\mathbf{s}}$. □

4.3. Third characterization: $\mathbf{RSU} = ((\mathbf{Q_0^+})^{\mathbf{sf}})_{[\mathbf{2}]}$

A key to the third characterization is the following

Lemma 4.2. $\mathcal{R}^{\mathbf{2}\times\mathbf{2}} \cap (\mathbf{Q_0^+})^{\mathbf{sf}} \subseteq \mathbf{P_0}$.

Proof. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a, d \geqslant 0$ and suppose $M \notin \mathbf{P_0}$, that is, $ad - bc < 0$.

Case i: $a + d > 0$. Then the 2×2 principal pivot on M yields

$$M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where at least one entry of the diagonal of M^{-1} is negative, so $M \notin (\mathbf{Q_0^+})^{\mathbf{f}}$.

Case ii: $a = d = 0$. Here $bc > 0$. If $b, c > 0$, then it can be easily verified that $M \notin \mathbf{Q_0}$. On the other hand, if $b, c < 0$, then SMS with $S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ has the two off diagonal entries positive, so $M \notin \mathbf{Q_0^f}$. □

Corollary 4.4. $\mathcal{R}^{\mathbf{2}\times\mathbf{2}} \cap (\mathbf{Q_0^+})^{\mathbf{sf}} = \mathcal{R}^{\mathbf{2}\times\mathbf{2}} \cap \mathbf{RSU}$.

Proof. We have

$$\mathcal{R}^{\mathbf{2}\times\mathbf{2}} \cap (\mathbf{Q}_\mathbf{0}^+)^{\mathbf{sf}} \subseteq \mathbf{P_0} \cap \mathbf{Q}_\mathbf{0}^\mathbf{f} \subseteq \mathbf{RSU}, \tag{26}$$

where the first inclusion is justified by Lemma 4.2 and by observing that $(\mathbf{Q}_\mathbf{0}^+)^{\mathbf{sf}} \subseteq \mathbf{Q_0}$; the second inclusion follows from Theorem 4.6.

On the other hand, since the diagonal entries of a $\mathbf{P_0}$ matrix are nonnegative, we have

$$\mathbf{RSU} \subseteq \mathbf{P_0} \cap \mathbf{Q_0} \subseteq \mathbf{Q}_\mathbf{0}^+$$

which (since **RSU** is both sign-change invariant and full) implies that

$$\mathbf{RSU} \subseteq (\mathbf{Q}_\mathbf{0}^+)^{\mathbf{sf}}. \tag{27}$$

Combining (26) and (27) we conclude that

$$\mathcal{R}^{\mathbf{2}\times\mathbf{2}} \cap (\mathbf{Q}_\mathbf{0}^+)^{\mathbf{sf}} = \mathcal{R}^{\mathbf{2}\times\mathbf{2}} \cap \mathbf{RSU}. \qquad \square$$

Theorem 4.8. $\mathbf{RSU} = ((\mathbf{Q}_\mathbf{0}^+)^{\mathbf{sf}})_{[\mathbf{2}]}$

Proof. The proof is easily obtained by considering Theorem 4.1 and Corollary 4.4. $\square$

From this characterization theorem, we obtain the

Corollary 4.5. $((\mathbf{Q}_\mathbf{0}^+)^{\mathbf{sf}})_{[\mathbf{2}]} = (\mathbf{Q}_\mathbf{0}^+)^{\mathbf{csf}}$.

Proof. By definition, $((\mathbf{Q}_\mathbf{0}^+)^{\mathbf{sf}})_{[\mathbf{2}]} \subseteq (\mathbf{Q}_\mathbf{0}^+)^{\mathbf{csf}}$. Now if $M \in ((\mathbf{Q}_\mathbf{0}^+)^{\mathbf{sf}})_{[\mathbf{2}]}$, then it must belong to **RSU**. Hence M and all its principal pivot transforms are sign-invariant and complete. Therefore $M \in (\mathbf{Q}_\mathbf{0}^+)^{\mathbf{csf}}$. $\square$

5. Conclusions

In this paper we have given three new characterizations of **RSU**, the class of row sufficient matrices. In the process, we have shown that $\mathbf{RSU} \subset \mathbf{L}$; this strengthens the main result of [1]. Our characterizations of **RSU** are expressed in terms of three structural properties (sign-change invariance, completeness, and fullness) and three other well known matrix classes: $\mathbf{L}$, $\mathbf{E_0}$, and $\mathbf{Q_0}$. A fourth structural property, reflectiveness, when coupled with the new characterizations of **RSU**, gives three new characterizations of $\mathbf{SU} = \mathbf{RSU}^\mathbf{r}$:

$$\mathbf{SU} = \mathbf{L}^{\mathbf{cfr}} = (\mathbf{E_0} \cap \mathbf{Q_0})^{\mathbf{sr}} = ((\mathbf{Q}_\mathbf{0}^+)^{\mathbf{sf}})_{[\mathbf{2}]})^\mathbf{r}.$$

In the course of establishing these characterizations, we have revealed a number of other interesting matrix class inclusions. It is conceivable that the application of structural properties such as those identified here will lead to better understanding of matrix classes in the study of the linear complementarity problem and other topics.

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