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# ARBITRAGE AND GROWTH RATE FOR RISKLESS INVESTMENTS IN A STATIONARY ECONOMY

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A sequential investment is a vector of payments over time, $(a_0, a_1, \ldots, a_n)$, where a payment is made to or by the investor according as $a_i$ is positive or negative. Given a collection of such investments it may be possible to assemble a portfolio from which an investor can get "something for nothing," meaning that without investing any money of his own he can receive a positive return after some finite number of time periods. Cantor and Lipmann (1995) have given a simple necessary and sufficient condition for a set of investments to have this property. We present a short proof of this result. If arbitrage is not possible, our result leads to a simple derivation of the expression for the long-run growth rate of the set of investments in terms of its "internal rate of return."

KEY WORDS: investment program, growth rate, no arbitrage, cash stream valuation, positive polynomial, convolution of vectors

## 1. INTRODUCTION

This paper deals with the situation of an agent, who we will refer to as an *investor*, who is able to choose from a given set, $\mathcal{S}$, of riskless financial actions, for example, saving and/or borrowing at some given interest rate, or taking out a loan which is to be paid off over a specified number of periods, etc. In fact we will consider the most general such action, given by any vector $\boldsymbol{a} = (a_0, a_1, \ldots, a_n)$ where the entry $a_i$ is the payment or *cash flow* in period $i$. A positive cash flow is a payment to the investor; a negative flow is a payment by the investor. We call any such cash flow vector a riskless *investment project*. Given an initial amount of money $m$ and a time horizon $T$, the most obvious problem for the investor is to choose projects in such a way as to maximize his cash on hand, $m_T$, in period $T$, the key fact being that payments to the investor from one project can be used to make payments into another. Although this is a straightforward linear programming problem, it may be quite complicated if there are many possible investments, and there is not much which can be said about the solution in general. However, Cantor and Lipmann (1995) have shown that for any such set of projects there is a unique well-defined asymptotic *growth factor*, $\bar{r}$, which describes the long-run growth of the quantity $m_T$. Namely, if $r$ is any positive number greater (less) than $\bar{r}$, then $m_T/r^T$ approaches zero (infinity) as $T$ goes to infinity. Further the authors give a simple explicit expression for $\bar{r}$ which is easily calculated from the set $\mathcal{S}$. Our purpose here is to give a short proof of a slightly weaker version of the Cantor-Lipmann (1995) result using a rather different approach. It turns out that for perfectly plausible sets of projects (see the examples below) it may happen that the maximum cash on hand $m_T$ at some finite period $T$ may be infinity, thus giving an "infinite" growth rate. This is

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equivalent to the possibility of what is called *arbitrage*, a subject which would seem to be of some interest in itself. Our approach is first to derive the simple necessary and sufficient conditions for arbitrage to be possible and then to obtain the growth theorem as an easy consequence. The key assumption which makes this analysis possible is that of *stationarity*, meaning that any project can be initiated in any time period. Using this property we show that the "Arbitrage Theorem" is equivalent to a purely algebraic result about polynomials.

Historically, the first result of this sort is the paper of Gale (1965), which considered the special case of a single $n$-period investment project in which the first $n-1$ terms are negative and the last term positive. For this case an explicit solution of the linear programming problem was found and the growth factor was the "internal rate of return." Cantor and Lipmann (1983) solved the problem for a single investment with no restrictions on the signs of the cash flows, followed by their solution of the general case in 1995.

Finally we note that models involving multiperiod riskless investments where the cash flows are unrestricted in sign have been considered in connection with other problems, the typical one being to attach a value to a given stream of cash flows. Recent examples are the papers of Dermody and Rockafellar (1991) and (1995). Although their problem is different from ours, our work has in common the important assumption that there is no externally given "term structure" which would be needed if one were to apply the classical criterion of net present value. Indeed, as we will point out in the next section, if one assumes, for example, that there is an interest rate $r$, the same for both borrowing and lending (so-called perfect capital market), then our problem becomes trivial and every investment project has growth rate $r$ or zero or infinity.

## 2. THE MODEL AND EXAMPLES

As in the introduction, a *riskless investment project*, $\boldsymbol{a}$, is a sequence of real numbers called *payments* (or *cash flows*), say, of money. Thus, $\boldsymbol{a} = (a_0, a_1, \ldots, a_n)$, where $a_i$ is the payment $i$ periods after the investment has been initiated. Defined in this general way, investments include sequences like $(1, -r)$ to be thought of a borrowing rather than investing, so the reader should be aware of this terminology.

We will be concerned with programs of investment over some finite number of time periods, and the assumption of *stationarity* states that any investment can be initiated in any time period.

For obvious reasons it will always be assumed that each investment project $\boldsymbol{a}$ has at least one positive and one negative entry, and we will normalize by assuming $a_0 = 1$ or $-1$. A two-period investment of the form $\boldsymbol{a} = (-1, r)$ ($\boldsymbol{a} = (1, -r)$) will be called *saving (borrowing) with interest factor* $r$. If $r = -1$ (1) we refer to *flat saving (borrowing)*.

We shall say that a set of investments *permits arbitrage* if it is possible by investing suitably to get "something for nothing," that is, without ever introducing money from the outside, to have a positive amount of money after a finite number of periods.[1]

An obvious example is the case where it is possible to borrow at rate $r$ and save (or lend) at rate $s$ with $s > r$. A second well-known but less obvious case occurs when there is some interest factor $r$ at which it is possible both to lend and to borrow. In our notation this means that the investments $(1, -r)$ and $(-1, r)$ are both available at all times. This is sometimes referred to as a "perfect capital market." Recall that the *present value*, $PV_{\boldsymbol{a}}(r)$

[1]Because of homogeneity, if one can get a positive amount of money, one can get an arbitrarily large amount of money by investing at a sufficiently high level, thus becoming "infinitely rich."

of the investment $\boldsymbol{a} = (a_0, a_1, \ldots, a_n)$ at interest factor $r$ is given by

$$PV_{\boldsymbol{a}}(r) = a_0 + a_1 r^{-1} + \cdots + a_n r^{-n}.$$

It is known that arbitrage is possible if and only if $PV_{\boldsymbol{a}}(r)$ is positive.

The case of perfect capital markets is very special and perhaps rather unrealistic since most investors, if they are able to borrow at all, probably find the available borrowing rate $s$ higher than the best available saving rate $r$. Indeed, available borrowing rates typically vary with the individual investor, depending, for example, on the amount of collateral the investor is able to put up. Nevertheless, in these cases arbitrage may still be possible and the condition turns out to be that an investment can achieve arbitrage if and only if it has a positive present value for all interest factors on the closed interval $[r, s]$. However, even if there is no such investment it may still be possible to achieve arbitrage by combining various investments. The general result, which will be proved in Section 3, is easily stated.

ARBITRAGE THEOREM. *Arbitrage is possible for a given set $\mathcal{S}$ of investments if and only if for any positive value of $r$ there is at least one investment in $\mathcal{S}$ whose present value at $r$ is positive.*

The Arbitrage Theorem has a simple graphical interpretation. From now on, given an investment $\boldsymbol{a} = (a_0, a_1, \ldots, a_n)$, we define the *investment polynomial*, $P_{\boldsymbol{a}}$, to be the polynomial $P_{\boldsymbol{a}}(\alpha) = a_0 + a_1\alpha + \cdots + a_n\alpha^n$. Thus the present value of $\boldsymbol{a}$ at interest factor $r$ is $P_{\boldsymbol{a}}(r^{-1})$.

Let $\mathcal{S}$ be given by $\{\boldsymbol{a}^1, \boldsymbol{a}^2, \ldots, \boldsymbol{a}^m\}$ and let $P_{\boldsymbol{a}^i}$ be the polynomial of $\boldsymbol{a}^i$. Then arbitrage is possible if and only if the "upper envelope," $\max_i[P_{\boldsymbol{a}^i}(\alpha)]$, lies above the $x$-axis. We denote this upper envelope by $A_{\mathcal{S}}(\alpha)$.

If arbitrage is not possible then the function $A_{\mathcal{S}}(\alpha)$ must be nonpositive for some values of $\alpha$. Let $\bar{\alpha}$ be the largest such value. We then have our second result, proved in Section 4.

GROWTH THEOREM. *The long-run growth factor (to be defined precisely later) of $\mathcal{S}$ is the reciprocal of $\bar{\alpha}$.*

Although the Arbitrage Theorem would seem to be both natural and basic, it does not appear to have been known up to the time of Cantor and Lipmann (1995). Perhaps this is because the condition of stationarity is essential. Further, it may be necessary to invest for many periods before obtaining the positive "payoff." The following examples indicate some of the things that can happen.

EXAMPLE 2.1. You receive \$1 today. You must pay \$2 tomorrow, and you then receive \$1.01 the day after, so $\boldsymbol{a} = (1, -2, 1.01)$, and hence the investment is just barely "profitable." It is assumed you have no money on hand to begin with so the only way you can pay a second day's installment on a unit investment is by initiating a second investment at level 2. It turns out, however, that after proceeding in this way for 32 periods you end up receiving a positive payment and not owing anything, thus achieving arbitrage. This should be contrasted with

EXAMPLE 2.2. You receive \$1 today, pay \$20 tomorrow, and receive \$100 the day after, so $\boldsymbol{a} = (1, -20, 100)$. Clearly this is a highly profitable investment since one sees that an "input" of 20 produces a return of 101, so one seems to be getting a return of better than 5 to

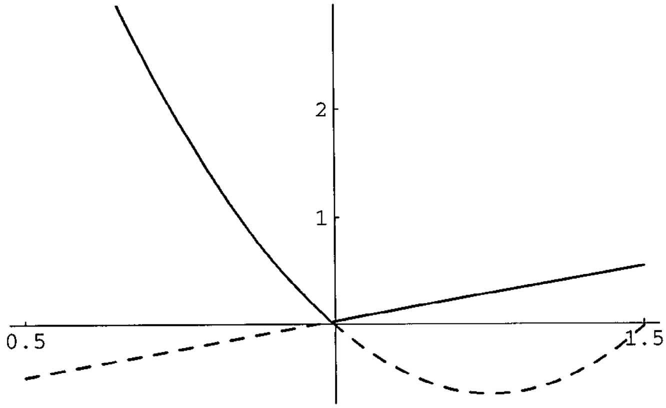


FIGURE 2.1.

1. Nevertheless, for this investment arbitrage is not possible, and indeed if one starts such an investment it will never be possible to get out of debt without bringing in money from the outside. (It will be shown that this investment actually has “long-run growth factor” 10 rather than the intuitive little over 5.)

Here are some examples involving more than one investment.

EXAMPLE 2.3. You receive \$15 today, pay \$25 tomorrow, but only get back \$9.99 on the third day. This is clearly a losing proposition. However, if in addition you can save with $r = 1.03$ (i.e., put money in a savings account which pays 3% interest), then by suitably combining investments you can realize a positive return after nine periods.

EXAMPLE 2.4. Though it may not be intuitively clear, once you start the investment (9, −12, 4) you will never be able to stop without still owing some money. However, if you throw in the clearly losing saving (−1, .75), then, as we shall demonstrate, you can realize a positive return (after 10 periods).

Figures 2.1 and 2.2 give the graphs of the polynomials associated with the last two examples. One can clearly see that while the necessary and sufficient condition of the Arbitrage Theorem is not satisfied by any one of the (four) individual investments, it is satisfied by the combination of the two investments in each of the examples.

The main virtue of the proofs given here as compared to those of Cantor and Lipmann (1995) is that they are considerably shorter and the result is slightly more general in that it does not assume that the set of possible investments must include flat savings. On the other hand, Cantor and Lipmann (1995) obtain tighter bounds in approximating the growth rate than the ones we get here.

## 3. THE ARBITRAGE THEOREM

It will be convenient to use the convention that an $n$-vector, $\boldsymbol{a}$, is an infinite sequence having entry $a_n$ nonzero and entries $a_k = 0$ for $k > n$. This allows us to add and compare vectors of different lengths.

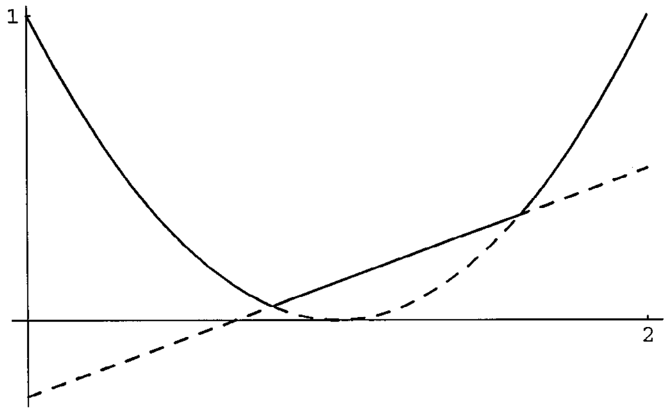


FIGURE 2.2.

Given an investment $\boldsymbol{a} = (a_0, a_1, \ldots, a_n)$, homogeneity and stationarity allow the investor to initiate investment $\boldsymbol{a}$ at any desired nonnegative level $x_t$ at any period $t = 0, 1, \ldots$ (think of $x_t$ as the number of "shares" of $\boldsymbol{a}$ acquired by the investor in period $t$). Given a nonnegative vector $\boldsymbol{x} = (x_0, x_1, \ldots, x_\ell)$, we denote by $\boldsymbol{z} = (z_0, z_1, \ldots, z_{n+\ell})$ the cash flow vector resulting from starting investment $\boldsymbol{a}$ at level $x_t$ $(t = 0, 1, \ldots, \ell)$. By direct computation we see that $z_t = x_0 a_t + x_1 a_{t-1} + \cdots + x_t a_0$.

This leads us to introduce the following basic device.

DEFINITION. The *convolution,* $\boldsymbol{u} \circ \boldsymbol{v}$, of $\boldsymbol{u} = (u_0, u_1, \ldots, u_m)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ is the $(m+n+1)$-vector $\boldsymbol{w}$ whose $k$th entry is $w_k = u_0 v_k + u_1 v_{k-1} + \cdots + u_k v_0$.

In terms of convolution the equation of the previous paragraph becomes simply $\boldsymbol{z} = \boldsymbol{x} \circ \boldsymbol{a}$. We now extend these constructs to the general case of several possible investments.

DEFINITIONS. Let $\mathcal{S} = \{\boldsymbol{a}^1, \boldsymbol{a}^2, \ldots, \boldsymbol{a}^m\}$ be a set of investments. An *investment program* $\boldsymbol{z}$ *of* $\mathcal{S}$ is defined as

$$\boldsymbol{z} = \boldsymbol{x}^1 \circ \boldsymbol{a}^1 + \boldsymbol{x}^2 \circ \boldsymbol{a}^2 + \cdots + \boldsymbol{x}^m \circ \boldsymbol{a}^m, \tag{3.1}$$

where $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^m$ is a set of nonnegative vectors. The program $\boldsymbol{z}$ is called *feasible* if it is nonnegative (feasibility corresponds to the requirement of not introducing money from the outside).

We call a vector $\boldsymbol{u}$ *positive,* $\boldsymbol{u} > 0$, if it has at least one positive entry and no negative entries. Thus, arbitrage is realized whenever $\boldsymbol{z}$ as given in (3.1) is positive. In fact, we can restate the **Arbitrage Theorem** as follows:

*There exist nonnegative vectors* $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^m$ *such that* $\boldsymbol{z} = \sum_i \boldsymbol{x}^i \circ \boldsymbol{a}^i$ *is positive if and only if* $A_{\mathcal{S}}(\alpha) = \max_i [P_{\boldsymbol{a}^i}(\alpha)]$ *is positive for all nonnegative* $\alpha$.

As stated in the introduction, we shall prove the Arbitrage Theorem by purely algebraic arguments about polynomials. To do it we need the following

KEY OBSERVATION. If $P_{\boldsymbol{u}}$, $P_{\boldsymbol{v}}$, and $P_{\boldsymbol{w}}$ are the polynomials corresponding to vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ then $P_{\boldsymbol{w}} = P_{\boldsymbol{u}} P_{\boldsymbol{v}}$ if and only if $\boldsymbol{w} = \boldsymbol{u} \circ \boldsymbol{v}$.

In other words, convolution and polynomial multiplication are the same thing. It follows that the convolution operation is commutative and associative. In view of the key observation, (3.1) is equivalent to

$$P_{\boldsymbol{z}} = P_{\boldsymbol{x}^1} P_{\boldsymbol{a}^1} + P_{\boldsymbol{x}^2} P_{\boldsymbol{a}^2} + \cdots + P_{\boldsymbol{x}^m} P_{\boldsymbol{a}^m}. \tag{3.2}$$

Let us call a polynomial *positive* if all of its coefficients are positive. The Arbitrage Theorem is then reduced to the following purely algebraic statement.

POSITIVE POLYNOMIAL LEMMA. *If $P_1, P_2, \ldots, P_m$ is a set of polynomials, there exist positive polynomials $Q_1, Q_2, \ldots, Q_m$ such that $\sum_i Q_i P_i$ is positive if and only if the upper envelope $A_{\mathcal{S}}(\alpha) = \max_i\{P_i(\alpha)\}$ is positive for all nonnegative $\alpha$.*

The proof of the lemma will be given in the appendix.

Note that in Example 2.1 it was possible to achieve arbitrage even without flat saving (though adding flat saving would reduce the period of waiting for arbitrage from 33 to 18 days). On the other hand, even if there is flat saving, an investment such as (100,−1) does not permit arbitrage. However, as an immediate consequence of our theorem we have

COROLLARY. *If the set $\mathcal{S}$ of investments includes flat saving, then it permits arbitrage if and only if $A_{\mathcal{S}}(\alpha)$ is positive on the interval* $(0, 1]$.

## 4. THE GROWTH THEOREM

We turn now to the case where arbitrage from a set of given investments $\mathcal{S}$ is not possible. To simplify definitions we assume that it is possible to transfer money from one period to the next; thus, flat saving $(-1, 1)$ is always available. From the corollary any nonpositive value of $A_{\mathcal{S}}(\alpha)$ must occur for $\alpha$ in $[0, 1]$.

As in the introduction we define $m_T$ to be the maximum amount of money which can be obtained in period $T$ from an input of 1 in period 0. Because of the presence of flat savings there is an equivalent, more convenient, way to define $m_T$. Let $\boldsymbol{z}$ be any $T$-period program for the set of investments $\mathcal{S}$. Since arbitrage is not possible, $\boldsymbol{z}$ must have at least one negative entry. Let $z^-$ and $z^+$ be the sum of the negative and positive entries of $\boldsymbol{z}$ respectively. We claim

$$m_T = \max -(z^+/z^-) \qquad \text{where } \boldsymbol{z} \text{ runs over all } T\text{-period programs.} \tag{4.1}$$

Namely, multiplying $\boldsymbol{z}$ by a constant if necessary, we may assume that $z^- = -1$. Then if 1 unit is available in period 0 it can be saved so as to make the required payments which are the negative entries of $\boldsymbol{z}$. Similarly the positive entries of $\boldsymbol{z}$ which sum to $m_T$ can be saved until the final period $T$.

We can now give a precise formulation of the

GROWTH THEOREM. *Let $\bar{\alpha}$ be the largest $\alpha$ for which $A_{\mathcal{S}}(\alpha) \le 0$ and let $\bar{r} = \bar{\alpha}^{-1}$. Then*

(i) *If $r > \bar{r}$, then $m_T < r^T$ for all $T$ (so $m_T/r^T \to 0$).*

(ii) *If* $r < \bar{r}$, *then there is a* $T_0$ *and* $k > 0$ *such that* $m_T > kr^T$ *for* $T > T_0$ *(so* $m_T/r^T \to \infty$*).*

We remark that Cantor and Lipmann (1995) give a somewhat stronger result than (ii), giving a tighter bound on the inequality.[2] The number $\bar{r}$ is called the *long run growth factor* of $\mathcal{S}$. The following example shows that the long run growth factor can behave in a highly discontinuous way with respect to the set of investments.

EXAMPLE 4.1. Suppose that if an investor pays $19 in the present period he can get an "interest-free loan" of $400 in the next period which must be repaid in the following period. Thus, $\boldsymbol{a} = (-19, 400, -400)$. Then $P_{\boldsymbol{a}}(\alpha)$ has roots .05 and .95. If it is also possible to save with a 5% interest rate, then the growth factor of the two investments is 1.05. If, however, the interest rate is 5.5%, then the growth factor jumps to 20.

*Proof of the Growth Theorem.* To prove (i), note that if $A_{\mathcal{S}}(\alpha) \leq 0$ then for any investment program $\boldsymbol{z}$, $P_{\boldsymbol{z}}(\alpha) \leq 0$, which follows from (3.2). Namely, each $P_{\boldsymbol{a}^i}(\alpha)$ is nonpositive; hence, so is $P_{\boldsymbol{x}^i} P_{\boldsymbol{a}^i}(\alpha)$ since $P_{\boldsymbol{x}^i}$ is nonnegative, so $P_{\boldsymbol{z}}(\alpha)$ is nonpositive, being the sum of nonpositive terms. Now let $\boldsymbol{r}$ be the $T$-vector $(-1, 0, \ldots, r^T)$ with $r > \bar{r}$. If $m_T > r^T$, this means there is a $T$-period program $\boldsymbol{z}$ such that $\boldsymbol{z} > \boldsymbol{r}$, and this would mean that $P_{\boldsymbol{z}} > P_{\boldsymbol{r}}$, but $P_{\boldsymbol{z}}(\bar{\alpha}) \leq 0$ and $P_{\boldsymbol{r}}(\bar{\alpha}) = r^T \bar{\alpha}^T - 1 = (r/\bar{r})^T - 1 > 0$, which gives a contradiction.

To prove (ii), suppose $r < \bar{r}$. Then adjoin to $\mathcal{S}$ the borrowing investment $\boldsymbol{b} = (1, -r)$. Note that $P_{\boldsymbol{b}}(\alpha) = 1 - r\alpha$ is positive for $\alpha < r^{-1}$. On the other hand, by hypothesis $A_{\mathcal{S}}(\alpha)$ is positive for $\alpha > \bar{\alpha} > r^{-1}$, so the set $\mathcal{S} \bigcup \boldsymbol{b}$, satisfies the hypothesis of the Arbitrage Theorem, so there exist some $T_0$ and an investment program with levels $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^n$ and $\boldsymbol{x}^0$ such that $\sum \boldsymbol{x}_i \circ \boldsymbol{a}^i + \boldsymbol{x}_0 \circ \boldsymbol{b} > 0$ or, letting $\boldsymbol{w} = \sum \boldsymbol{x}^i \circ \boldsymbol{a}^i$,

$$\boldsymbol{w} > -\boldsymbol{x}^0 \circ \boldsymbol{b} = \boldsymbol{x}^0 \circ (-1, r). \tag{4.2}$$

We now take the convolution of both sides of (4.2) with the positive $T$-vector $\boldsymbol{r} = (1, r, \ldots, r^T)$, where $T > T_0$. By direct computation, $(-1, r) \circ \boldsymbol{r} = (-1, 0, \ldots, r^{T+1})$, and the convolution of this with $\boldsymbol{x}^0$ gives a vector whose first $T_0 + 1$ entries are $-\boldsymbol{x}^0$ and whose last $T_0 + 1$ entries are $r^{T_0+1}\boldsymbol{x}^0$, and all other entries are zero. We claim the desired investment program is $\boldsymbol{z} = \boldsymbol{w} \circ \boldsymbol{r}$, for we have

$$\boldsymbol{z} > (-\boldsymbol{x}^0, 0, 0, \ldots, r^T \boldsymbol{x}^0). \tag{4.3}$$

The right side of (4.3) is a $(T + N)$-vector, where $N$ is the length of $\boldsymbol{x}_0$. Using (4.1) we see that $m_{T+N} > r^T$, or if $T' = N + T$ then $m'_T > r^{T'-N} = kr^{T'}$, where $k = r^{-N}$. □

## 5. REMARKS

It is natural to ask whether arbitrage opportunities like the ones given by our examples actually exist in real life. Most of the interesting examples involve investment in which

[2] Specifically, it is shown that there are positive constants $c_1 < c_2$ such that $m_T$ satisfies the inequalities $c_1/T^h \bar{\alpha}^T < m_T < c_2/T^h \bar{\alpha}^T$, where $h$ is the right-hand order of the root $\bar{\alpha}$. (We say that a function has right-hand order $h$ at $x$ if the function and its first $h - 1$ right-hand derivatives vanish but its $h$th derivative does not.)

there are two or more sign changes in the sequence of payments, and such sequences seem to be uncommon. The more typical situation is either a bond in which the investor makes the initial payment and receives the subsequent interest or a mortgage in which the investor receives an initial amount and pays it off over time. Second, the condition of stationarity may not generally be satisfied. Riskless investments which are available today may not be available tomorrow. For example, obviously, bond prices fluctuate. In view of these observations, the question of whether there are practical applications of the Arbitrage Theorem would require further empirical investigation.

## APPENDIX

Here again is the statement of the

POSITIVE POLYNOMIAL LEMMA. *Let $P_1, P_2, \ldots, P_m$ be a set $S$ of polynomials. Then there exist positive polynomials $Q_1, Q_2, \ldots, Q_m$ such that $\sum_i Q_i P_i$ is positive if and only if the upper envelope $A_{\mathcal{S}}(\alpha) = \max_i\{P_i(\alpha)\}$ is positive for all nonnegative $\alpha$.*

The necessity is trivially proved by observing that $A_{\mathcal{S}}(\alpha) < 0$ for some nonnegative $\alpha$ implies that $\sum_i Q_i(\alpha)P_i(\alpha) < 0$ for any positive polynomials $Q_1, Q_2, \ldots, Q_m$.

Now we turn our attention to sufficiency. The case $n = 1$ is a theorem of Poincaré (1883) and can be phrased as follows.

POINCARÉ THEOREM. *If $P$ is polynomial positive on $R^+$, there exists a positive polynomial $Q$ such that $QP$ is positive.*

In fact, given the polynomial $P$, Poincaré gives an explicit expression for the positive polynomial $Q$.

The proof of the lemma for general $n$ is due to George Bergman (personal communication). For the inductive argument to follow we shall need

LEMMA A.1. *Given positive numbers $a, b, p, q$ with $a < b$, there exist a positive integer $n$ and a positive number $\theta$ such that $\theta x^n \leq p$ on $[0, a]$ and $\theta x^n \geq q$ on $[b, \infty]$.*

*Proof.* Choose $n$ so that $(b/a)^n > q/p$. Then choose $\theta = p/a^n$ and verify that the conclusion holds. □

LEMMA A.2. *Given $0 \leq a < b < c$ and polynomials $P$ and $P'$, where $P$ is positive on $[a, c]$ and $P'$ is positive on $[b, \infty]$. Then there exist positive polynomials $Q$ and $Q'$ such that $QP + Q'P'$ is positive on $[a, \infty]$.*

*Proof. Case I. $P$ and $P'$ are bounded below.* Choose $\delta > 0$ such that $-\delta$ is a lower bound for $P$ and $P'$, and choose positive numbers $\beta$ and $\gamma$ such that $P > \beta$ on $[a, c]$ and $P' > \gamma$ on $[b, \infty]$. Let $Q = 1$ and $Q' = \theta x^n$, where (Lemma A.1) $\theta x^n < \beta/\delta$ on $[0, c]$ and $\theta x^n > \delta/\gamma$ on $[b, \infty]$. Then

- On $[a, c]$, $P + \theta x^n P' > \beta + (\beta/\delta)(-\delta) = 0$.
- On $[b, c]$, both $P$ and $P'$ are positive so $P + \theta x^n P' > 0$.
- On $[c, \infty]$, $P + \theta x^n P' > -\delta + (\delta/\gamma)\gamma = 0$.

*Case II. $P$ is unbounded below.* Let $d$ be its degree. Since $P'$ is positive on $[b, \infty]$ its leading coefficient is positive. Thus, $P''(x) = P(x)+\theta x^d P'(x)$ is unbounded above, hence bounded below. Further, by taking $\theta$ sufficiently small we can ensure that $P''$ is positive on $[a, c]$. Now apply Case I to $P''$ and $P'$. □

*Proof of the Positive Polynomial Lemma—sufficiency.* For each polynomial let $\mathcal{X}_i$ be the set of points where $P_i$ is positive. Each $\mathcal{X}_i$ consists of a finite set of open intervals, and the hypothesis that $A_S(x)$ is positive means that these intervals cover $R^+$. The case of a single interval is the Poincaré Theorem. For the general case, suppose $\mathcal{X}_1$ is the leftmost unbounded interval and $\mathcal{X}_2$ is a finite interval which overlaps it on the left. Then by Lemma A.2 there is a positive polynomial combination $P'$ of $P_1$ and $P_2$ which is positive on the union of $\mathcal{X}_1$ and $\mathcal{X}_2$. Adjoining $P'$ to the set of polynomials we have reduced the number of intervals by one. Continuing in this way we eventually get a polynomial which is positive on all of $R^+$, and this can again be made positive by the Poincaré Theorem. □

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