# A direct reduction of PPAD Lemke-verified linear complementarity problems to bimatrix games

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#### Abstract

The linear complementarity problem, LCP(q, M), is defined as follows. For given  $M \in$  $\mathcal{R}^{m \times m}, q \in \mathcal{R}^m$ , find z such that  $q + Mz \geq 0, z \geq 0, z^{\intercal}(q + Mz) = 0$ , or certify that there is no such z. It is well known that the problem of finding a Nash equilibrium for a bimatrix game (2-NASH) can be formulated as a linear complementarity problem (LCP). In addition, 2-NASH is known to be complete in the complexity class PPAD (Polynomial-time Parity Argument Directed). However, the ingeniously constructed reduction (which is designed for any PPAD problem) is very complicated, so while of great theoretical significance, it is not practical for actually solving an LCP via 2-NASH, and it may not provide the potential insight that can be gained from studying the game obtained from a problem formulated as an LCP (e.g. market equilibrium). The main goal of this paper is the construction of a simple explicit reduction of any LCP(q, M)that can be verified as belonging to  $\mathcal{PPAD}$  via the graph induced by the generic Lemke algorithm with some positive covering vector d, to a symmetric 2-NASH. In particular, any endpoint of this graph (with the exception of the initial point of the algorithm) corresponds to either a solution or to a so-called secondary ray. Thus, an LCP problem is verified as belonging to  $\mathcal{PPAD}$  if any secondary ray can be used to construct, in polynomial time, a certificate that there is no solution to the problem. We achieve our goal by showing that for any M, q and positive d satisfying a certain nondegeneracy assumption with respect to M, we can simply and directly construct a symmetric 2-NASH whose Nash equilibria correspond one-to-one to the end points of the graph induced by LCP(q, M) and the Lemke algorithm with a covering vector d. We note that for a given M the reduction works for all positive d with the exception of a subset of measure 0.

### 1 Introduction

The linear complementarity problem LCP(q, M) is defined as

For given  $q \in \mathcal{R}^m$ ,  $M \in \mathcal{R}^{m \times m}$ , find  $z \in \mathcal{R}^m$  such that  $q + Mz \ge 0$ ,  $z \ge 0$ ,  $z^{\mathsf{T}}(q + Mz) = 0$ .

The LCP is notable for its wide range of applications, from well understood and relatively easy to solve problems, such as linear and convex quadratic programming problems, to  $\mathcal{NP}$ -hard problems. A

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major effort in LCP theory had been the study of variants of Lemke's algorithm, a Simplex-like vertex following algorithm. In particular, for a given positive covering vector d, the Lemke(d) algorithm goes through a path of adjacent vertices of the 'extended' LCP(q, M) (denoted by ELCP(d, q, M)) where d is attached to M with an artificial variable  $z_0$ . Assuming (without loss of generality) that ELCP(d, q, M) is nondegenerate, Lemke(d) is guaranteed to terminate in a finite number of steps with either a solution to the original problem or with a secondary ray of ELCP(d, q, M). If the secondary rays can certify (in polynomial time) that there is no solution to LCP(q, M), we say that the problem is Lemke(d)-resolvable. One of the major themes of LCP research over the years has been the search for classes of matrices M and covering vectors d for which LCP(q, M) is Lemke(d)resolvable for all q. Several such classes (usually applicable for all d > 0) were identified (see e.g. [CPS92], [Mur88] and the references therein).

The introduction of the  $\mathcal{PPAD}$  (Polynomial-time Parity Argument Directed) complexity class in [Pap94] provides an effective and elegant framework for analyzing the complexity of Lemke(d)resolvable linear complementarity problems since, in general, the directed graph induced by the Lemke(d) algorithm for a given LCP(q, M) can be used to verify the membership of the problem in  $\mathcal{PPAD}$ . We say in this case that the problem is  $Lemke(d) \mathcal{PPAD}$ -verified. This development is significant with respect to LCP theory since it has been shown in [MP91] that if  $\mathcal{PPAD}$  is  $\mathcal{NP}$ -hard then  $\mathcal{NP} = Co\mathcal{NP}$ , lending support to the long standing informal belief that LCPs resolvable by Lemke(d) algorithm are in some way special.

What makes the class  $\mathcal{PPAD}$  particularly interesting is the fact that several well known problems, such as finding a Brouwer fixed-point, were identified in [Pap94] as  $\mathcal{PPAD}$ -complete. The discovery, in a string of papers ([DP05], [DP05a], [CD05] and [CD05a]), that finding a Nash equilibrium of a bimatrix game (2-NASH) is  $\mathcal{PPAD}$ -complete has significant consequences in the context of LCP theory. It has been known since the early days of LCP research that the 2-NASH problem can be formulated as an LCP with roughly the same size and with the coefficient matrix belonging to one of several well known classes resolvable by Lemke(d) algorithm. The fact that 2-NASH is  $\mathcal{PPAD}$ -complete means that any LCP(q, M) verifiable as a member in  $\mathcal{PPAD}$  (including all classes that contain 2-NASH) can be reduced to a 2-NASH problem. However, the known reduction is quite complicated. It requires several stages that involve reducing the given LCP(q, M) to finding an approximate Brouwer fixed point of an appropriate function, followed by reducing the latter to 3-graphical NASH (using small polymatrix games to simulate the computation of certain simple arithmetic operations), and finally, reducing the 3-graphical NASH to 2-NASH<sup>1</sup>. While there seems to be no discussion in the vast literature on LCP suggesting the possibility that Lemke(d)  $\mathcal{PPAD}$ verified LCPs can be reduced to 2-NASH, the discovery that 2-NASH is  $\mathcal{PPAD}$ -complete motivated us to search for the existence of a direct simple reduction of such problems to 2-NASH.

The main result of this paper is the introduction of a direct, simple reduction of almost any Lemke(d)  $\mathcal{PPAD}$ -verified linear complementarity problems to a symmetric 2-NASH. In fact, we introduce a stronger result as follows. Consider a generic Lemke(d) LCP(q, M) (which we call LLCP(d, q, M)) whose 'solutions' are defined to be either actual solutions of LCP(q, M) or secondary rays of ELCP(d, q, M). Obviously this problem<sup>2</sup> is Lemke(d)  $\mathcal{PPAD}$ -verified. Through a series of steps we show how to construct a symmetric bimatrix game whose equilibria correspond one-to-one

<sup>&</sup>lt;sup>1</sup>A clear 'bird's-eye view' description of the reduction can be found in [DGP09].

<sup>&</sup>lt;sup>2</sup>Where, as we assume without loss of generality, its extended form, ELCP(d, q, M), is nondegenerate.

to the 'solutions' of LLCP(d, q, M). The point is that if LCP(q, M) is Lemke(d)  $\mathcal{PPAD}$ -verified, the Nash equilibria of the constructed bimatrix game correspond to solutions (or certifications for infeasibility) of the given problem, which means that the constructed bimatrix game properly resolves an LCP(q, M) if it is Lemke(d)-verified.

We begin by reviewing the Lemke(d) algorithm (in Section 2) and bimatrix games (in Section 3). Next, we introduce (in Section 4) the complexity class  $\mathcal{PPAD}$ , and briefly discuss the notion of Lemke(d)  $\mathcal{PPAD}$ -verified LCPs. In addition, we present the majority of matrix classes known to be Lemke(d)  $\mathcal{PPAD}$ -verified, and identify a number of matrix classes whose corresponding LCPs are  $\mathcal{PPAD}$ -complete. We conclude the introductory sections by introducing (in Section 5) LLCP(d, q, M), the generic Lemke(d) LCP(q, M).

Our main results are presented in sections 6-8. We start by introducing in Section 6 a very simple reduction of LCP(q, M), where M belongs to a class of matrices for which a solution is guaranteed to exist for all q, to a symmetric 2-NASH. The cost matrix of the resulting bimatrix game is composed of M with an extra row and column. In particular, we show that the solutions of the given LCP(q, M) correspond one-to-one to the Nash equilibria which use with positive probability for the pure strategy corresponding to the extra column of the cost matrix. Moreover, we show that the Nash equilibria which do not use the pure strategy corresponding to the extra column of the so-called 'secondary directions' of ELCP(e, q, M). Note that at this stage we address only e - the vector of all ones - as a covering vector, and that we do not reach yet our goal as the reduction may produce secondary directions rather than secondary rays.<sup>3</sup>

In Section 7, we extend the basic reduction above (by considering an augmented problem) so that the constructed bimatrix game produces either a solution for LCP(q, M), a secondary ray for ELCP(e, q, M), or a non-zero vector which is a solution to LCP(e, M) (and is actually also a special case of a secondary direction of ELCP(e, q, M)).

In Section 8, we show that if a secondary direction generated by the bimatrix game constructed in the previous section is a nondegenerate solution of LCP(e, M), we can use it to compute, in strongly polynomial time either a solution for LCP(q, M) or a secondary ray for ELCP(e, q, M); thus showing that the constructed bimatrix game indeed provides a 'solution' for LLCP(e, q, M).

In Section 9 we extend the results of the previous section to accommodate the reduction of any LLCP(d, q, M) for which d > 0 and LCP(d, M) is nondegenerate; thereby achieving our goal of showing that any LCP(q, M) which is Lemke $(d) \mathcal{PPAD}$ -verified (satisfying our nondegeneracy assumption as stated above) can be reduced to a symmetric 2-NASH. We note that for any given M and q, the reduction is workable for all positive covering vectors with the exception of a finite number of sets of measure 0.

The constructed reduction is particulary useful since it provides a bijection between the reducible LCPs and their corresponding 2-NASH problems. In particular, the simplicity of the reduction and its bijection property allows for the practical use of the results of the extensive research on 'non Lemke type' 2-NASH algorithms for solving (or enumerating the solutions of) reducible LCPs . In addition, these reductions can be applied to investigate properties of solutions of reducible LCP via known properties of the associated 2-NASH problems. We discuss these subjects together with

<sup>&</sup>lt;sup>3</sup>A ray of ELCP(d, q, M) is an unbounded edge of ELCP(d, q, M) which includes its endpoint (a vertex of ELCP(q, M)) together with a direction vector.

additional concluding remarks in Section 10.

Throughout the paper we denote by e vectors all of whose entries are 1. Given a matrix A, we denote by  $A_i$ , the i-th row of A, by  $A_{.j}$  the j-th column of A, and by  $A_{ij}$  the ij-th entry of A. We denote by  $\mathcal{R}^{m \times n}, \mathcal{R}^{m \times n}_+$ , and  $\mathcal{R}^{m \times n}_{++}$  the space of  $m \times n$  real matrices, the space of nonnegative  $m \times n$  real matrices, and the space of positive  $m \times n$  real matrices, respectively. Whenever n = 1 we abbreviate  $\mathcal{R}^{m \times n}$  to  $\mathcal{R}^m$ , and whenever m = n = 1 we abbreviate  $\mathcal{R}^{m \times n}$  to  $\mathcal{R}$ .

### 2 LCP and Lemke's algorithm

Given  $M \in \mathcal{R}^{m \times m}$ ,  $q \in \mathcal{R}^m$ , the linear complementarity problem, LCP(q, M), is defined as

find 
$$z \in \mathcal{R}^m$$
 such that

$$q + Mz \ge 0, \quad z \ge 0, \tag{1a}$$

$$z^{\mathsf{T}}(q+Mz) = 0. \tag{1b}$$

Note that (1a)–(1b) imply

$$z_i(q_i + M_i, z) = 0, \quad i = 1, \dots, m.$$
 (1c)

We denote by FEA(q, M) the set of all z satisfying (1a), and by SOL(q, M) the set of all z satisfying (1a) and (1b).

In this section we present the generic Lemke algorithm (the so-called Scheme I - see [CPS92], 4.4.5). Given LCP(q, M) we define its *extended* version, with a *covering vector* d > 0 as

$$ELCP(d, q, M) \triangleq \{z_0 \in \mathcal{R}_+, z \in \mathcal{R}_+^m \mid q + dz_0 + Mz \ge 0 \text{ and } z^{\mathsf{T}}(q + dz_0 + Mz) = 0\}.$$

Note that ELCP(d, q, M) is composed of a polyhedral set intersected with one nonlinear complementarity constraint. Throughout the paper whenever we refer to vertices, edges and rays of ELCP(e, q, M) we mean the vertices, edges and rays of the polyhedral set associated with ELCP(e, q, M). We assume that ELCP(d, q, M) is nondegenerate, that is that the polyhedral set associated with it is nondegenerate<sup>4</sup>. Let  $(\bar{z}_0, \bar{z}) \in ELCP(d, q, M), \bar{w} = q + d\bar{z}_0 + M\bar{z}$ , and let kbe the the number of positive entries in  $(\bar{z}_0, \bar{z}, \bar{w})$ . By the nondegeneracy assumption, k is equal to either m (in which case  $(\bar{z}_0, \bar{z})$  is a vertex of ELCP(d, q, M)), or m + 1 (in which case it is a point on an edge of ELCP(d, q, M)). If a vertex of ELCP(d, q, M) is contained in an edge, we say that the vertex is an *endpoint* of the edge. If an edge of ELCP(d, q, M) is unbounded then it corresponds to a ray of ELCP(d, q, M), which can be presented as

$$\left\{ \begin{bmatrix} z_0 \\ z \end{bmatrix} \mid \begin{bmatrix} z_0 \\ z \end{bmatrix} = \begin{bmatrix} \bar{z}_0 \\ \bar{z} \end{bmatrix} + \begin{bmatrix} \bar{u}_0 \\ \bar{u} \end{bmatrix} \lambda \text{ for all } \lambda \ge 0 \right\}$$
(2)

where

$$(\bar{z}_0, \bar{z})$$
 is a vertex of  $ELCP(d, q, M)$ ,  
 $\bar{u} \in SOL(d\bar{u}_0, M), \ \bar{u}_0 \in \{0, 1\}$  and  $(\bar{u}_0, \bar{u}) \neq 0$ ,  
 $\bar{z}^{\mathsf{T}}(d\bar{u}_0 + M\bar{u}) = 0$  and  $\bar{u}^{\mathsf{T}}(q + d\bar{z}_0 + M\bar{z}) = 0$ .

<sup>&</sup>lt;sup>4</sup>There is no loss of generality in this assumption since if it is not satisfied, we perturb q by applying standard linear programming techniques.

Consider the ray of ELCP(d, q, M) with  $\bar{z} = 0$ ,  $\bar{z}_0 = -\min_{1 \le i \le m} q_i$ ,  $\bar{u}_0 = 1$ ,  $\bar{u} = 0$ . We call this ray the *primary ray*, and its corresponding endpoint vertex the *initial vertex*. Any other ray of ELCP(d, q, M), can be characterized as

$$(\bar{z}_0, \bar{z})$$
 is a vertex of  $ELCP(d, q, M)$ , (4a)

$$\bar{u} \in SOL(d\bar{u}_0, M) \setminus \{0\}, \ \bar{u}_0 \in \{0, 1\} \text{ and } e^{\mathsf{T}}\bar{u} = 1 \text{ whenever } \bar{u}_0 = 0,$$
 (4b)

$$\bar{z}^{\mathsf{T}}(d\bar{u}_0 + M\bar{u}) = 0 \text{ and } \bar{u}^{\mathsf{T}}(q + d\bar{z}_0 + M\bar{z}) = 0.$$
 (4c)

We call such a ray, a secondary ray. We denote the set of all secondary rays of ELCP(d, q, M) by

$$SR(d,q,M) \triangleq \{(\bar{z}_0, \bar{z}, \bar{u}_0, \bar{u}) \text{ satisfying } (4a) - (4c)\}.$$

Note that a secondary ray  $(\bar{z}_0, \bar{z}, \bar{u}_0, \bar{u})$  has two components, a vertex  $(\bar{z}_0, \bar{z})$  of ELCP(d, q, M) and what we call a secondary direction  $(\bar{u}_0, \bar{u})$  as defined in (4b). Specifically, we denote the set of all secondary directions of ELCP(d, q, M) as

$$SD(d, M) \triangleq \{(\bar{u}_0, \bar{u}) \text{ satisfying (4b)}\}.$$

We distinguish between two types of secondary directions (and rays), according to whether  $\bar{u}_0 = 0$ , which we call a *type* 0 secondary direction, or  $\bar{u}_0 = 1$ , which we call a *type* 1 secondary direction. Specifically, for k = 0, 1, we denote the set of all type k secondary directions of ELCP(d, q, M) by

$$SD_k(d, M) \triangleq \{ (\bar{u}_0, \bar{u}) \in SD(d, q, M) \mid \bar{u}_0 = k \}.$$

Similarly, for k = 0, 1, we we denote the set of all type k secondary rays of ELCP(d, q, M) by

$$SR_k(d,q,M) \triangleq \{(\bar{z}_0, \bar{z}, \bar{u}_0, \bar{u}) \in SR(d,q,M) \mid \bar{u}_0 = k\}.$$

Starting with the initial vertex of ELCP(d, q, M), the generic Lemke(d) algorithm traces a unique<sup>5</sup> finite path of adjacent vertices of ELCP(d, q, M), terminating with either a solution to LCP(q, M) or with a secondary ray of ELCP(d, q, M). Specifically, the algorithm ends with either a vertex  $(\bar{z}_0, \bar{z})$  of ELCP(d, q, M) with  $\bar{z}_0 = 0$  (so  $\bar{z} \in SOL(q, M)$ ), or with a secondary ray  $(\bar{z}_0, \bar{z}, \bar{u}_0, \bar{u}) \in SR(d, q, M)$ with  $\bar{z}_0 > 0$ . We say that Lemke(d) resolves a given LCP(q, M) if either it ends with  $\bar{z} \in SOL(q, M)$ , or if the terminal secondary ray can certify that  $SOL(q, M) = \emptyset$ . Whenever Lemke(d) resolves LCP(q, M) we say that LCP(q, M) is Lemke(d)-resolvable.

Ever since the introduction of the Lemke algorithm [Lem65], extensive research efforts focused on identifying classes of matrices M for which LCP(q, M) is Lemke(d)-resolvable for all q. In the following we discuss two major groups of matrices containing almost all known classes of matrices M for which LCP(d, q, M) is Lemke(d)-resolvable for all q.

The first group is based on the idea that if  $SR(d,q,M) = \emptyset$ , then Lemke(d) outputs  $\overline{z} \in SOL(q,M)$ . Specifically, we consider the class of d-regular matrices (see [CPS92], 3.9.20) as defined below.

**Definition** Given  $M \in \mathcal{R}^{m \times m}$  and  $d \in \mathcal{R}^{m}_{++}$ , we say that M is *d*-regular if  $SOL(d\tau, M) = \{0\}$  for all  $\tau \in \mathcal{R}_{+}$ . We denote the class of *d*-regular matrices by  $\mathbf{R}(d)$ .

<sup>&</sup>lt;sup>5</sup>The uniqueness is due to the assumption that ELCP(d, q, M) is nondegenerate.

It follows that if  $M \in \mathbf{R}(d)$ , then for all q, ELCP(d, q, M) has no secondary directions and thus no secondary rays. That is,  $SR(d, q, M) = \emptyset$  for all q. Recalling that Lemke(d) terminates in a finite number of steps with either  $\overline{z} \in SOL(q, M)$  or with  $(\overline{z}_0, \overline{z}, \overline{u}_0, \overline{u}) \in SR(d, q, M)$ , we conclude that whenever  $M \in \mathbf{R}(d)$ , LCP(q, M) is Lemke(d)-resolvable for all q.

**Remark**<sup>6</sup> It is well known that M belongs to the *strictly semimonotone* matrix class (which is denoted by **E**) if and only if  $SOL(q, M) = \{0\}$  for all  $q \ge 0$  (see [CPS92],3.9.11). Thus, it follows that for all  $d > 0, \mathbf{E} \subset \mathbf{R}(d)$ . In addition, **E** properly contains the *strictly copositive* matrix class (**C**), and the class of all matrices whose principle minors are positive (**P**). Thus, we observe that LCP(q, M) with M in **E**, **C** or **P** is Lemke(d)-resolvable for all d > 0 and all q.

The second group includes classes of matrices for which  $SR(d, q, M) \neq \emptyset$  implies that  $SOL(q, M) = \emptyset$ . Specifically, most matrix classes with this property that have been identified in the LCP literature share the following property:

$$SD_1(d,M) = \emptyset$$
 (5a)

$$SR_0(d, q, M) \neq \emptyset \Rightarrow FEA(q, M) = \emptyset.$$
 (5b)

We denote by **USR**(d) (for useful secondary rays) the class of all matrices M for which (5a)-(5b) are satisfied for all  $q \ge 0$ .

The class of matrices that satisfy (5a) is defined below.

**Definition** Given  $M \in \mathcal{R}^{m \times m}$  and  $d \in \mathcal{R}^{m}_{++}$ , we say that  $M \in \mathcal{R}^{m \times m}$  is *d-semiregular* if  $SOL(d, M) = \{0\}$ . We denote the class of *d*-semiregular matrices by  $\mathbf{R}_0(d)$ .

#### Remarks

- 1. While the term 'd-semiregular' is introduced here for the first time, the class itself has been introduced in [Gar73] under the name  $\mathbf{E}^*(d)$ .
- 2. If  $M \in \mathbf{USR}(d)$  then the existence of a secondary ray for ELCP(d, q, M) implies that  $SOL(q, M) = \emptyset$ . Hence, any LCP(q, M) with  $M \in \mathbf{USR}(d)$  is Lemke(d)-resolvable for all q.
- 3.  $\mathbf{R}(d) \subset \mathbf{USR}(d)$ .
- 4. It is well known that M belongs to the *semimonotone* matrix class (which is denoted by  $\mathbf{E}_0$ ) if and only if  $SOL(q, M) = \{0\}$  for all q > 0 (see [CPS92],3.9.3). Thus, it follows that  $\mathbf{E}_0 \subset \mathbf{R}_0(d)$ for all d > 0. In addition,  $\mathbf{E}_0$  properly includes the *copositive* matrix class ( $\mathbf{C}_0$ ), and the class of all matrices whose principle minors are nonnegative ( $\mathbf{P}_0$ ).
- 5. There are two well known classes of matrices,  $\mathbf{L}$  and  $\mathbf{Q_0} \cap \mathbf{P_0}$ , which are known to be in  $\mathbf{USR}(d)$  for all d > 0. In particular, major matrix classes, including *Column Sufficient* ( $\mathbf{CSU}$ ), *Row Sufficient* ( $\mathbf{RSU}$ ), and *Sufficient* ( $\mathbf{SU}$ ), are subsets of  $\mathbf{P_0} \cap \mathbf{Q_0}$ , while *Copositive Plus* ( $\mathbf{C_0^+}$ ), and *Copositive Star* ( $\mathbf{C_0^*}$ ) are subsets of  $\mathbf{L}$ . Hence, LCP(q, M) where M belongs to any of these classes of matrices is Lemke(d)-resolvable. For a discussion of these and other Lemke(d)-resolvable classes see [CPS92] and [Mur88]. Figure 1 at the end of Section 5 depicts the relationship among these classes.

<sup>&</sup>lt;sup>6</sup>The definitions of all the matrix classes which are mentioned in this paper can be found in [CPS92].

### **3** Bimatrix Games

Let  $A, B \in \mathcal{R}^{m \times n}$  be the cost matrices of the row and column players of a bimatrix game. A Nash equilibrium of this game is a pair of vectors  $x \in \mathcal{R}^m$ ,  $y \in \mathcal{R}^n$  (representing mixed strategies for the row and column players respectively), satisfying

$$Ay \ge e(x^{\intercal}Ay), \ B^{\intercal}x \ge e(x^{\intercal}By), \ e^{\intercal}x = e^{\intercal}y = 1, \ x \ge 0, \ y \ge 0.$$

To simplify the presentation we restrict our attention to symmetric bimatrix games where  $A = B^{\intercal}$ . In particular, it has been shown in the seminal paper [Nas51] that every symmetric bimatrix game has a symmetric Nash equilibrium (that is, a Nash equilibrium where x = y). In addition, it is well known that the Nash equilibria for any bimatrix game with cost matrices A, B (which can be assumed, without loss of generality, to be positive) can be easily extracted from the symmetric equilibria of the symmetric bimatrix game with cost matrix  $\begin{pmatrix} 0 & A \\ B^{\intercal} & 0 \end{pmatrix}$ .

Given  $C \in \mathcal{R}^{n \times n}$ , we denote by SG(C) the symmetric bimatrix game where the row and column players' cost matrix is C. We say that  $x \in \mathcal{R}^n$  is a symmetric Nash equilibrium of SG(C) if

$$Cx \ge e(x^{\mathsf{T}}Cx),\tag{6a}$$

$$x \ge 0,\tag{6b}$$

$$e^{\mathsf{T}}x = 1. \tag{6c}$$

Note that since  $x^{\intercal}Cx = \sum_{i=1}^{m} x_i(C_i, x)$ , (6a)–(6b) imply

$$x_i(C_i \cdot x - x^{\mathsf{T}}Cx) = 0, \quad i = 1, \dots, n.$$
 (9d)

We denote by SNE(C) the set of symmetric Nash equilibria of SG(C). We refer to the problem of finding a symmetric Nash equilibrium for SG(C) as solving SG(C).

There are several ways of formulating the problem of finding a Nash equilibrium of a bimatrix game as a linear complementarity problem ([CD68], [Eav71], [MZ91], [Sav06]). Here we adopt the reduction in [Sav06], where the problem of computing a symmetric Nash equilibrium of a symmetric bimatrix game is presented as a linear complementarity problem. In particular, let C be the cost matrix of a symmetric bimatrix game. Without loss of generality we can assume (by adding a sufficiently large constant to all the entries of C) that C > 0. Solving SG(C) with C > 0 can be reduced to LCP(-e, C) as described in [Sav06], and presented in the following theorem.

**Theorem 1** Suppose C > 0.

- (i) Let  $z \in SOL(-e, C)$ . Then,  $z \frac{1}{e^{\intercal} z} \in SNE(C)$ .
- (ii) Let  $x \in SNE(C)$ . Then,  $x \frac{1}{x^{\intercal}Cx} \in SOL(-e, C)$ .

**Proof.** (i) and (ii) can be easily verified by substitution.

### 4 The Complexity Class PPAD

The class  $\mathcal{PPAD}$  (*Polynomial-time Parity Argument Directed*), which was introduced in the seminal paper [Pap94], is a class of problems which can be presented as follows.

**Definition** Given a directed graph with every node having in-degree and out-degree at most one described by a polynomial-time computable function f(v) that outputs the predecessor and successor of a node v, and a node s (which we call the *initial source node*) with a successor but no predecessor, find a node  $t \neq s$  which is either a *sink* (a node with no successor) or a *source* (a node with no predecessor), but not both. We call such a graph the  $\mathcal{PPAD}$  graph associated with the problem.

Many important problems, such as the Brouwer fixed-point problem, the search versions of Smith's theorem, the Borsuk-Ulam theorem and, as previously discussed, Nash equilibrium of bimatrix game, belong to this class [Pap94]. Interestingly, the problems in  $\mathcal{PPAD}$  are generally believed not to be  $\mathcal{NP}$ -hard since it has been shown in [MP91] that if there exists a  $\mathcal{PPAD}$  problem which is  $\mathcal{NP}$ -hard then  $\mathcal{NP} = \mathcal{CoNP}$ . What makes the study of this class attractive is that it has been shown that several problems within the class (such as the Brouwer fixed-point problem) are  $\mathcal{PPAD}$ -complete with strong circumstantial evidence that these problems are not likely to have a polynomial time algorithm [HPV89].

The  $\mathcal{PPAD}$  complexity class seems to be a natural framework for analyzing the computational complexity of Lemke(d)-resolvable LCP(q, M), as the underlying graph of Lemke(d) whose nodes correspond to the vertices and edges of ELCP(d, q, M) has a structure reminiscent of a  $\mathcal{PPAD}$  graph. In particular, given ELCP(d, q, M), we define its associated graph (which we call the Lemke(d) graph associated with LCP(q, M)), as the directed graph G(d, q, M) whose nodes correspond to the vertices and edges (including rays) of ELCP(d, q, M). There is an arc (u, v) of G(d, q, M) if and only if either u corresponds to a vertex of ELCP(d, q, M), v corresponds to an edge of ELCP(d, q, M) and the vertex corresponding to u is the tail of the edge corresponding to v; or u corresponds to an edge of ELCP(d,q,M), v corresponds to a vertex of ELCP(d,q,M) and the vertex corresponding to v is the head of the edge corresponding to v. The orientations of the edges are determined according the scheme presented in [Tod76]. We identify the node associated with the primary ray of ELCP(d, q, M)as the required special source node of a  $\mathcal{PPAD}$  graph. Given (as we assume) that ELCP(d, q, M)is nondegenerate, we have that every node of G(d,q,M) is incident to at most two edges, and that there are no isolated nodes. Thus, G(d, q, M) is a nonempty collection of simple directed paths. In addition, any node incident to only one other node (except for the node associated with the primary ray) corresponds to either a solution of LCP(q, M) or to a secondary ray of ELCP(d, q, M). Thus, if for a given LCP(q, M) and a covering vector d, the secondary rays of ELCP(d, q, M) can certify (in polynomial time in the size of LCP(q, M) that  $SOL(q, M) = \emptyset$ , we can conclude that LCP(q, M)is in  $\mathcal{PPAD}$ . Whenever this is the case, we say that LCP(q, M) is  $Lemke(d) \mathcal{PPAD}$ -verified.

Indeed, in [Pap94], one of the first examples of a  $\mathcal{PPAD}$  problem is an LCP(q, M) where  $M \in \mathbf{P}$ . While it is customary in the literature of linear complementarity to discuss methods for solving LCP(q, M) under the assumption that M possesses some special properties, it creates difficulties from an algorithmic complexity point of view, as verifying these properties may be by itself a hard problem (e.g. identifying a  $\mathbf{P}$  matrix is  $\mathcal{CoNP}$  complete [Cox73]). Thus, in [Pap94], the problem at hand (which is called  $\mathbf{P} - LCP$ ) is defined as follows. Given q, M, either find  $z \in SOL(q, M)$ , or provide a certificate (with size polynomial in the size of the problem) for  $M \notin \mathbf{P}$ . Motivated by the discussion in [Pap94] we consider the following generic problem.

 $\mathbf{Y} - LCP(q, M)$ : Given  $M \in \mathcal{R}^{m \times m}$ ,  $q \in \mathcal{R}^m$  and a matrix class  $\mathbf{Y}$ , find one of the following:

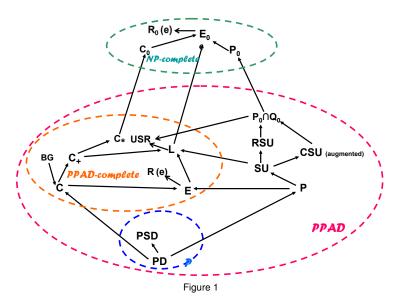
(1)  $z \in SOL(q, M)$ , (2) a certificate that  $SOL(q, M) = \emptyset$ , (3) a certificate that  $M \notin \mathbf{Y}$ .

We say that  $\mathbf{Y} - LCP(q, M)$  is Lemke(d)-PPAD-verified if any  $(\bar{z}_0, \bar{z}, \bar{u}_0, \bar{u}) \in SR(d, q, M)$  leads (in polynomial time in the size of ELCP(d, q, M)) to (2) or (3) above.

**Remark** Following the discussion in section 2, we have that  $\mathbf{USR}(q, M) - LCP(q, M)$  is Lemke(d)  $\mathcal{PPAD}$ -verified and for all q and d > 0. In particular, considering the remarks at the end of Section 2, we can conclude that  $\mathbf{L} \cup (\mathbf{P_0} \cap \mathbf{Q_0}) - LCP(q, M)$  is Lemke(d)  $\mathcal{PPAD}$ -verified and for all q and d > 0.

### 5 The generic Lemke(d) linear complementarity problem

As stated in the introduction, it has been established that the problem of finding a Nash equilibrium for a bimatrix game is  $\mathcal{PPAD}$ -complete. Moreover, since solving any bimatrix game is polynomially reducible to solving a symmetric bimatrix game, we have that the problem of finding a symmetric Nash equilibrium for a symmetric bimatrix game, as presented in Section 3, is also  $\mathcal{PPAD}$ -complete. In particular, it is shown there that this problem can be represented as an LCP(-e, M) where M > 0. Since M > 0 implies that  $M \in \mathbb{C}$  (the class of all matrices for which  $0 \neq x \in \mathcal{R}^m_+$  implies that  $x^{\mathsf{T}}Mx > 0$ ), and considering the remark at the end of the previous section, we conclude that  $\mathbb{C} - LCP$  is  $\mathcal{PPAD}$ -complete as well. In Figure 1, we display the relationship among the classes of matrices discussed in previous sections. An arrow from a class  $\mathbf{X}$  to a class  $\mathbf{Y}$  indicates that  $\mathbf{X} \subset \mathbf{Y}$ . So for any class  $\mathbf{Y}$  reachable by a directed path from class  $\mathbb{C}$  in Figure 1 we have that if  $\mathbf{Y} - LCP$ is in  $\mathcal{PPAD}$  then it is  $\mathcal{PPAD}$ -complete. Note that the class  $\mathbf{USR}(d)$  (for any d > 0) contains all the classes of matrices  $\mathbf{Y}$  identified in the previous section as a classes for which  $\mathbf{Y} - LCP(q, M)$  is Lemke(d)  $\mathcal{PPAD}$ -verified and for all q and d > 0.



Next we show how Lemke(d)  $\mathcal{PPAD}$ -verified linear complementarity problems can be reduced simply and directly to a symmetric bimatrix game. To achieve this goal, we shall consider the following generic problem which we call Lemke(d)-LCP(q,M) and denote by LLCP(d,q,M).

LLCP(d,q,M): Given  $M \in \mathcal{R}^{m \times m}$ ,  $q \in \mathcal{R}^m$ , and  $d \in \mathcal{R}^m_{++}$ , find either  $z \in SOL(q,M)$  or  $(\bar{z}_0, \bar{z}, \bar{u}_0, \bar{u}) \in SR(d,q,M)$ .

Obviously LLCP(d, q, M) is verified to be in  $\mathcal{PPAD}$  by G(d, q, M). In the following we shall show how it is possible to reduce any LLCP(d, q, M), whenever LCP(d, M) is nondegenerate, to a symmetric bimatrix game. In addition, given  $\overline{d} > 0$ , we shall show, by standard LP perturbation techniques, that our reduction works for all d in a sufficiently small neighborhood of  $\overline{d}$ , and that the reduction works for all  $d \in \mathcal{R}_{++}$ , except for a finite number of subsets of measure zero. We shall present our reduction in several steps, where we address the reduction of instances of the following problem:

 $\mathbf{Y} - LLCP(d, q, M)$ : Given  $M \in \mathcal{R}^{m \times m}$ ,  $q \in \mathcal{R}^m$ ,  $d \in \mathcal{R}^m_{++}$  and a matrix class  $\mathbf{Y}$ , find one of the following.

(1)  $z \in SOL(q, M)$ , (2)  $(\bar{z}_0, \bar{z}, \bar{u}_0, \bar{u}) \in SR(d, q, M)$ , (3) A certificate that  $M \notin \mathbf{Y}$ .

We start by presenting in Section 6 a simple reduction which is applicable to  $\mathbf{R}(e) - LLCP(e, q, M)$ . The certificate that we obtain from the bimatrix game whenever  $M \notin \mathbf{R}(e)$  is of the form  $0 \neq \bar{u} \in SOL(e\bar{u}_0, M)$ . So we get either a solution for LCP(q, M) or a secondary direction for ELCP(e, q, M). Note that at this stage we address only e as a covering vector, and that we do not reach yet our goal as the reduction may produce secondary *directions* rather than secondary *rays*. In Section 7, we extend the previous reduction to handle  $\mathbf{R}_0(e) - LLCP(e, q, M)$ . The certificate that we obtain from the bimatrix game whenever  $M \notin \mathbf{R}_0(e)$  is of the form  $0 \neq \bar{u} \in SOL(e, M)$ . So we get either a solution for LCP(q, M), a secondary ray for ELCP(e, q, M), or a type 1 secondary direction for ELCP(e, q, M). Next, in Section 8, we present a complete reduction of LLCP(e, q, M) under the assumption that LCP(e, M) is nondegenerate if  $SOL(e, M) \neq \{0\}$ . Finally, in Section 9, we extend the reductions in the previous sections to a general covering vector d > 0 for which LCP(d, M) is nondegenerate if  $SOL(d, M) \neq \{0\}$ .

## 6 Reducing R(e) - LCP(q, M) to a symmetric bimatrix game

In this section we present a simple direct reduction of  $\mathbf{R}(e) - LCP(e, q, M)$ . In particular, given q, M, we construct a symmetric bimatrix game whose symmetric Nash equilibrium points correspond oneto-one to either  $\bar{z} \in SOL(q, M)$  or a certificate for  $M \notin \mathbf{R}(e)$  in the form of  $0 \neq \bar{u} \in SOL(e\bar{u}_0, M)$ where  $\bar{u}_0 \in \{0, 1\}$ , so  $(\bar{u}_0, \bar{u}) \in SD(e, M)$ .

Given LCP(q, M) with  $M \in \mathcal{R}^{m \times m}$ ,  $q \in \mathcal{R}^m$  and a covering vector e, we set n = m + 1 and a symmetric bimatrix game whose cost matrix C(q, M) is

$$C(q,M) \triangleq \left[ \begin{array}{cc} M & q+e \\ 0 & 1 \end{array} \right].$$
(7)

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Given C(q, M) as above, we denote any symmetric equilibrium point  $x \in SNE(C)$  as  $x \triangleq \begin{pmatrix} y \\ t \end{pmatrix}$ , where  $y \in \mathcal{R}^m$  and  $t \in \mathcal{R}$ . Given SNE(C(q, M)), we partition it to

$$SNE_+(C(q,M)) \triangleq \left\{ \left[ \begin{array}{c} y\\ t \end{array} \right] \in SNE(C(q,M)) \mid t > 0 \right\},$$

and

$$SNE_0(C(q, M)) \triangleq \left\{ \begin{bmatrix} y \\ t \end{bmatrix} \in SNE(C(q, M)) \mid t = 0 \right\},$$

In the next theorem we establish a one-to-one correspondence between the symmetric Nash equilibria of SG(C(q, M)) which use with positive probability the last column of C(q, M), and the set of solutions to LCP(q, M). We follow this with a theorem that establishes a one-to-one correspondence between the symmetric Nash equilibria of G(C(q, M)) which are not using the last column of C(q, M) and the secondary directions of ELCP(e, q, M).

#### Theorem 2

(i) Given 
$$\bar{x} = \begin{bmatrix} \bar{y} \\ \bar{t} \end{bmatrix} \in SNE_+(C(q, M)), let \ \bar{z} = \bar{y}\frac{1}{\bar{t}}.$$
 Then,  $\bar{z} \in SOL(q, M).$   
(ii) Given  $\bar{z} \in SOL(q, M), let \ \bar{t} = \frac{1}{e^{\tau}\bar{z}+1}, \ \bar{y} = \bar{z}\bar{t}.$  Then,  $\bar{x} = \begin{bmatrix} \bar{y} \\ \bar{t} \end{bmatrix} \in SNE_+(C(q, M)).$ 

**Proof.** Throughout the proof we denote C(q, M) by C.

- (i) Since  $\bar{t} > 0$ , then, by (9d) (with i = n),  $\bar{t} = \bar{x}^{\intercal}C\bar{x}$ . Thus, by (6a)–(6b),  $M\bar{y} + (q + e)\bar{t} \ge e\bar{t}$ ,  $\bar{y} \ge 0$ . Dividing by  $\bar{t}$ , we get  $M\bar{z} + q \ge 0$ ,  $\bar{z} \ge 0$ . In addition, by (9d) (for  $i = 1, \ldots, m$ ),  $\bar{y}_i(M_i.\bar{y} + q_i\bar{t} + \bar{t} \bar{t}) = 0$ , so dividing by  $\bar{t}^2$ , substituting for  $\bar{z}$ , and summing over m, we get  $0 = \sum_{i=1}^m \bar{z}_i(M_i.\bar{z} + q_i) = \bar{z}^{\intercal}(M\bar{z} + q)$ .
- (ii) By (1a), and setting  $\bar{t} = \frac{1}{e^{\tau}z+1}$ ,  $\bar{y} = \bar{z}\bar{t}$ , we have

$$\begin{bmatrix} M & q+e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{z}\bar{t} \\ \bar{t} \end{bmatrix} \ge \begin{pmatrix} e \\ 1 \end{pmatrix} \begin{pmatrix} \bar{z}\bar{t} \\ \bar{t} \end{pmatrix} \ge \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and obviously  $(e^{\dagger}\bar{z})\bar{t} + \bar{t} = (e^{\dagger}\bar{z} + 1)\bar{t} = 1$ . In addition,

$$\bar{x}^{\mathsf{T}}C\bar{x} = \bar{y}^{\mathsf{T}}(M\bar{y} + q\bar{t} + e\bar{t}) + \bar{t}^2 = \bar{t}^2(\bar{z}^{\mathsf{T}}(M\bar{z} + q) + e) + 1)$$

Thus, since by (1b),  $\bar{z}^{\intercal}(M\bar{z}+q) = 0$ ,  $\bar{x}^{\intercal}C\bar{x} = \bar{t}^2(\bar{z}^{\intercal}e+1) = \bar{t}$ . Hence,  $x = \begin{pmatrix} \bar{y} \\ \bar{t} \end{pmatrix}$  satisfies (6a)–(6c), and since  $\bar{t} > 0$ , we have  $\bar{x} \in SNE_+(C(q,M))$ .

(i) If 
$$\begin{bmatrix} \bar{y} \\ 0 \end{bmatrix} \in SNE_0(C(q, M))$$
, then  $\bar{y} \in SOL(e\tau, M)/\{0\}$  for some  $\bar{\tau} \ge 0$ .  
(ii) Let  $\bar{u} \in SOL(e\bar{\tau}, M)/\{0\}$  for some  $\bar{\tau} \ge 0$ . Then, setting  $\bar{y} = \bar{u}\frac{1}{e^{\tau}\bar{u}}, \bar{x} = \begin{bmatrix} \bar{y} \\ 0 \end{bmatrix} \in SNE_0(C(q, M)).$ 

**Proof.** Throughout the proof we denote C(q, M) by C.

- (i) By (6a)–(6b),  $M\bar{y} \ge e(\bar{y}^{\mathsf{T}}M\bar{y}), \ 0 \ge \bar{y}^{\mathsf{T}}M\bar{y}, \ \bar{y} \ge 0$ , and  $e^{\mathsf{T}}\bar{y} = 1$ . Setting  $\bar{\tau} = -\bar{y}^{\mathsf{T}}M\bar{y}$ , we get  $M\bar{y} + e\bar{\tau} \ge 0, \ 0 \ne \bar{y} \ge 0$ . Moreover, since  $e^{\mathsf{T}}\bar{y} = 1$ , we have (by (9d)) that  $\bar{y}^{\mathsf{T}}(M\bar{y} + e\bar{\tau}) = 0$ , concluding that  $\bar{y} \in SOL(e\bar{\tau}, M)/\{0\}$  for some  $\bar{\tau} \ge 0$ .
- (ii) Noticing that  $\bar{u} \neq 0$  and by (1a)–(1b),

$$M\bar{y} \ge e(-\bar{\tau}), \ \bar{y}^{\mathsf{T}}(M\bar{y} + e(-\bar{\tau})) = 0, \ \bar{y} \ge 0.$$

Thus,  $\begin{bmatrix} \bar{y} \\ 0 \end{bmatrix}$  satisfy (6a)–(6b). Noticing that  $e^{\mathsf{T}}\bar{u} = 1$ , completes the proof.  $\Box$ 

Given LCP(q, M), and combining Theorems 2 and 3, we can construct a symmetric bimatrix game where any symmetric Nash equilibrium point corresponds to either a solution for LCP(q, M), or a secondary direction for ELCP(e, q, M). Specifically, given  $q \geq 0$  and M, consider the symmetric bimatrix game whose cost matrix is C(q, M). Let  $\begin{bmatrix} \bar{y} \\ \bar{t} \end{bmatrix} \in SNE(C(q, M))$ , and let  $\bar{\tau}$  be its expected cost. We then conclude that:

**1**  $\overline{t} > 0$ . By Theorem 2-(i),  $\frac{1}{t}\overline{y} \in SOL(q, M)$ ,

**2**  $\bar{t} = \mathbf{0}$ . By Theorem 3-(i),  $\bar{y} \in SOL(e\bar{\tau}, M) \setminus \{0\}$  for some  $\bar{\tau} \ge 0$ ,

- **1.1**  $\bar{\tau} = \mathbf{0}$ . Then,  $\bar{y} \in SOL(0, M)$  with  $e^{\mathsf{T}}\bar{y} = 1$ , so  $\bar{y} \in SD_0(d, q, M)$ ,
- **1.2**  $\bar{\tau} > \mathbf{0}$ , Then,  $\bar{y}_{\bar{\tau}}^1 \in SOL(e, M)$ , so  $\bar{y}_{\bar{\tau}}^1 \in SD_1(e, q, M)$ .

So the symmetric bimatrix game SG(C) generates either a solution for LCP(q, M) or a secondary direction for ELCP(e, q, M).

#### Remarks

- 1. Note that  $\mathbf{R}(e) LLCP(e, q, M)$  is also reduced to SG(C(q, M)).
- 2. The class of all matrices M for which LCP(q, M) is guaranteed to have a solution for all q is called **Q**. The largest known class **Y** which is contained in **Q** and for which it is known that  $\mathbf{Y} LCP(q, M)$  is Lemke(e)-resolvable, is  $\mathbf{R}(e)$ .
- 3. Since  $\mathbf{P}, \mathbf{C} \subset \mathbf{E} \subset \mathbf{R}(e)$ , the reduction is applicable to  $\mathbf{Y} LCP(q, M)$ , where  $\mathbf{Y}$  is  $\mathbf{P}, \mathbf{C}$  or  $\mathbf{E}$ . Note that LCP(q, M) has a unique solution for all q if and only if  $M \in \mathbf{P}$ , and that LCP(q, M) has a unique solution for all  $q \geq 0$  if and only if  $M \in \mathbf{E}$ .

### 7 Reducing $R_0(e) - LLCP(e, q, M)$ to a symmetric bimatrix game

In this section we consider  $\mathbf{R}_0(e) - LLCP(e, q, M)$  which brings us closer to achieving our goal of reducing any LLCP(e, q, M) to a symmetric bimatrix game. In particular, given q, M, we construct a symmetric bimatrix game whose symmetric Nash equilibrium points correspond one-to-one to either the solutions of LCP(q, M), the secondary rays of ELCP(e, q, M), or certificates for  $M \notin \mathbf{R}_0(e)$  in the form of  $0 \neq \bar{u} \in SOL(e, M)$  which correspond to a type 1 secondary directions of ELCP(e, q, M).

For that purpose we introduce the *augmented* problem  $LCP(\tilde{q}, \tilde{M})$  associated with LCP(q, M), where

$$\tilde{M} = \left[ \begin{array}{cc} 1 & -e^{\mathsf{T}} \\ e & M \end{array} \right], \; \tilde{q} = \left[ \begin{array}{c} \beta \\ q \end{array} \right],$$

and  $\beta > e^{\mathsf{T}}\bar{z}$  for any vertex (not necessarily feasible)  $(\bar{z}_0, \bar{z})$  of ELCP(e, q, M).

#### Remarks

- 1. It is a standard result in LP theory that if the entries in q, M are rational then  $\beta$  is of size polynomial in the size of LCP(q, M), and that  $\beta$  can be computed in time polynomial in the size of LCP(q, M).
- 2. Augmented LCP systems where  $\tilde{M}_{11}$  is equal to 0 (see [CPS92]) or -1 [Tod73] are used in the LCP literature to eliminate secondary rays. Such augmentations do not work in our case since the reduction of  $LCP(\tilde{q}, \tilde{M})$  to a symmetric bimatrix game would yield a pure Nash equilibrium (using with probability 1 the strategy corresponding to the first column of  $\tilde{M}$ ) which yields no information about the solution (or lack thereof) of the original LCP(q, M). To avoid this possibility, we need  $\tilde{M}_{11} > 0$ , hence the choice of 1.

In the following theorem we establish the relationship between LCP(q, M) and  $LCP(\tilde{q}, \tilde{M})$ .

#### Theorem 4

- (i)  $\begin{bmatrix} 0\\ \bar{z} \end{bmatrix} \in SOL(\tilde{q}, \tilde{M}) \text{ if and only if } \bar{z} \in SOL(q, M).$
- (ii) If  $\begin{bmatrix} \tilde{z}_0 \\ \tilde{z} \end{bmatrix} \in SOL(\tilde{q}, \tilde{M})$  where  $\tilde{z}_0 > 0$ , then there exists  $(\bar{z}_0, \bar{z}, \bar{u}_0, \bar{u}) \in SR(e, q, M)$  and  $\bar{\lambda} > 0$  such that

$$\left[\begin{array}{c} \tilde{z}_0\\ \tilde{z} \end{array}\right] = \left[\begin{array}{c} \bar{z}_0\\ \bar{z} \end{array}\right] + \bar{\lambda} \left[\begin{array}{c} \bar{u}_0\\ \bar{u} \end{array}\right].$$

(iii) Let  $(\bar{z}_0, \bar{z}, \bar{u}_0, \bar{u}) \in SR(e, q, M)$ . Then there exists  $\bar{\lambda}$  such that  $\begin{bmatrix} \bar{z}_0 \\ \bar{z} \end{bmatrix} + \bar{\lambda} \begin{bmatrix} \bar{u}_0 \\ \bar{u} \end{bmatrix} \in SOL(\tilde{q}, \tilde{M})$ .

#### Proof.

(i) The 'only if' direction is obviously true. The 'if' direction is true because of the nondegeneracy assumption (so  $\tilde{z}$  is a vertex of LCP(q, M)) and by the definition of  $\beta$ .

(ii) Let  $\begin{bmatrix} \tilde{z}_0 \\ \tilde{z} \end{bmatrix} \in SOL(\tilde{q}, \tilde{M})$  where  $\tilde{z}_0 > 0$ . Then,  $q + e\tilde{z}_0 + M\tilde{z} \ge 0, \ \tilde{z} \ge 0, \ \tilde{z}_0 \ge 0, \ \tilde{z}^{\mathsf{T}}(q + e\tilde{z}_0 + M\tilde{z}) = 0, \ \tilde{z}_0(\beta + \tilde{z}_0 - e^{\mathsf{T}}\tilde{z}) = 0,$ 

which implies that  $(\tilde{z}_0, \tilde{z}) \in ELCP(e, q, M)$  and (since  $\tilde{z}_0, > 0$ )  $e^{\mathsf{T}}\tilde{z} = \beta + \tilde{z}_0$ . However, by the definition of  $\beta$ ,  $(\tilde{z}_0, \tilde{z})$  must be a point on a secondary ray of ELCP(e, q, M). That is, there exists  $\bar{\lambda} > 0$  and  $(\bar{z}_0, \bar{z}, \bar{u}_0, \bar{u}) \in SR(e, q, M)$  such that  $\begin{bmatrix} \tilde{z}_0 \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} \bar{z}_0 \\ \bar{z} \end{bmatrix} + \bar{\lambda} \begin{bmatrix} \bar{u}_0 \\ \bar{u} \end{bmatrix}$ .

(iii) By the definition of SR(e,q,M), we have that for all  $\lambda \ge 0$ ,  $\begin{bmatrix} \bar{z}_0 \\ \bar{z} \end{bmatrix} + \lambda \begin{bmatrix} \bar{u}_0 \\ \bar{u} \end{bmatrix}$  satisfies all the constraints of  $LCP(\tilde{q}, \tilde{M})$  except possibly for the last constraint. However, since  $-\bar{z}_0 + e^{\mathsf{T}}\bar{z} < \beta$ ,  $\bar{u} \ne 0, \bar{u}_0 \in \{0, 1\}$ , and  $-\bar{u}_0 + e^{\mathsf{T}}\bar{u} > 0$ , setting  $\bar{\lambda} = \frac{\beta + \bar{z}_0 - e^{\mathsf{T}}\bar{z}}{-\bar{u}_0 + e^{\mathsf{T}}\bar{u}}$  yields  $-(\bar{z}_0 + \bar{\lambda}\bar{u}_0) + e^{\mathsf{T}}(\bar{z} + \bar{\lambda}\bar{u}) = \beta$  which, considering that  $\bar{\lambda} > 0$ , completes the proof.

**Remark** The extraction of either a solution or a secondary ray from a secondary direction as described in the proof of Theorem 4-(ii) can be done by standard LP technique that can be executed in strongly polynomial time (that is, the required number of elementary calculations such as additions, multiplications, divisions and comparisons is bounded above by a polynomial function of m).

Next, we show that  $M \notin \mathbf{R}(e)$  implies that  $M \notin \mathbf{R}_0(e)$ , which allows us to apply the reduction of the previous section to the augmented problem.

**Theorem 5** If 
$$0 \neq \begin{bmatrix} \bar{u}_0 \\ \bar{u} \end{bmatrix} \in SOL(e\tau, \tilde{M})$$
 for some  $\tau \ge 0$ , then  $\bar{u}_0 + \bar{\tau} > 0$  and  $\frac{1}{\bar{u}_0 + \bar{\tau}} \bar{u} \in SOL(e, M)$ .

**Proof.** By the premise of the theorem there exists  $\bar{u}_0 \ge 0$ ,  $\bar{u} \ge 0$ ,  $\bar{u}_0 + e^{\dagger}\bar{u} > 0$ , such that

$$\begin{bmatrix} 1 & -e^{\mathsf{T}} \\ e & M \end{bmatrix} \begin{bmatrix} \bar{u}_0 \\ \bar{u} \end{bmatrix} + \begin{bmatrix} 1 \\ e \end{bmatrix} \bar{\tau} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ and}$$
$$[\bar{u}_0 \ \bar{u}^{\mathsf{T}}] \left( \begin{bmatrix} 1 & -e^{\mathsf{T}} \\ e & M \end{bmatrix} \begin{bmatrix} \bar{u}_0 \\ \bar{u} \end{bmatrix} + \begin{bmatrix} 1 \\ e \end{bmatrix} \bar{\tau} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus,

 $\bar{u}_0 - e^{\mathsf{T}}\bar{u} + \bar{\tau} \ge 0,\tag{8a}$ 

$$M\bar{u} + e(\bar{u}_0 + \bar{\tau}) \ge 0, \ \bar{u} \ge 0,$$
 (8b)

$$\bar{u}^{\mathsf{T}}(M\bar{u} + e(\bar{u}_0 + \bar{\tau})) = 0.$$
 (8c)

By (8a) we have that  $\bar{u}_0 + \bar{\tau} > 0$  (as otherwise  $\bar{u} = 0, \bar{u}_0 = 0$  contrary to the assumption). Thus, from (8b) and (8c) we have that  $\frac{1}{\bar{u}_0 + \bar{\tau}} \bar{u} \in SOL(e, M)$ .

Combining Theorems 4 and 5, and recalling the definition of the class  $\mathbf{R}_0(e)$ , we get that, given LCP(q, M), we can construct a symmetric bimatrix game where any symmetric Nash equilibrium point corresponds to a solution of LCP(q, M), a secondary ray of ELCP(e, q, M) or a type 1 direction of ELCP(e, q, M).

Specifically, consider the symmetric bimatrix game whose cost matrix is  $C(\tilde{q}, \tilde{M}) = \begin{bmatrix} 1 & -e^{\intercal} & -\beta + 1 \\ e & M & q + e \\ 0 & 0 & 1 \end{bmatrix}$ .

Let 
$$\begin{bmatrix} \bar{z}_0 \\ \bar{z} \\ \bar{t} \end{bmatrix} \in SNE(C(\tilde{q}, \tilde{M}))$$
, and let  $\bar{\tau}$  be its expected cost. We then conclude that:

- **1**  $\bar{t} > \mathbf{0}$ . By Theorem 2,  $\begin{bmatrix} \bar{z}_0 \\ \bar{z} \end{bmatrix} \frac{1}{\bar{t}} \in SOL(\tilde{q}, \tilde{M})$ .
  - **1.1**  $\bar{z}_0 = 0$ . Then (by Theorem 4-(i)),  $\bar{z}_t^1 \in SOL(q, M)$  (a solution to the original problem).
  - **1.2**  $\bar{z}_0 > 0$ . Then (by Theorem 4-(ii) and the remark following its proof), we can obtain (in strongly polynomial time)  $(\bar{z}_0, \bar{z}, \bar{u}_0, \bar{u}) \in SR(e, q, M)$  (a secondary ray of ELCP(e, q, M)).

**2** 
$$\bar{t} = \mathbf{0}$$
. By Theorem 3.  $0 \neq \begin{bmatrix} \bar{u}_0 \\ \bar{u} \end{bmatrix} \in SOL(e\bar{\tau}, \tilde{M})$  where  $\bar{\tau} \ge 0$ . Thus, by Theorem 5,  
 $0 \neq \bar{u} \frac{1}{\bar{u}_0 + \bar{\tau}} \in SOL(e, M)$  (so  $\begin{bmatrix} 1 \\ \bar{u} \frac{1}{\bar{u}_0 + \bar{\tau}} \end{bmatrix} \in SD_1(e, q, M)$  (a type 1 secondary direction of  $ELCP(e, q, M)$ ).

### 8 Handling nondegenerate type 1 secondary directions

In this section we show that if  $0 \neq \bar{u} \in SOL(e, M)$  is nondegenerate, then we can compute, in strongly polynomial time,  $(\bar{z}_0, \bar{z})$ , a vertex of ELCP(q, M) such that if  $\bar{z}_0 > 0$  then  $(\bar{z}_0, \bar{z}, 1, \bar{u}) \in SR_1(e, q, M)$  (that is, a type 1 secondary ray).

Let  $\bar{u}$  be a non-zero, nondegenerate solution for LCP(e, M). Setting  $\bar{v} = M\bar{u} + e$ , let  $\alpha = \{i \mid \bar{u}_i > 0\}$  and  $\bar{\alpha} = \{i \mid \bar{v}_i > 0\}$ . Note that by the nondegeneracy assumption  $\alpha \cup \bar{\alpha} = \{1, \ldots, m\}$  and  $M_{\alpha\alpha}$  is nonsingular. Now, we set

$$\begin{bmatrix} \hat{z}_{\alpha} \\ \hat{w}_{\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} M_{\alpha\alpha} & 0 \\ M_{\bar{\alpha}\alpha} & I \end{bmatrix}^{-1} \begin{bmatrix} q_{\alpha} \\ q_{\bar{\alpha}} \end{bmatrix}, \begin{bmatrix} \hat{z}_{\bar{\alpha}} \\ \hat{w}_{\alpha} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which results in

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$$\hat{w} = q + M\hat{z}, \ \hat{z}^{\mathsf{T}}\hat{w} = 0, \ \bar{u}^{\mathsf{T}}(q + M\hat{z}), \ \hat{z}^{\mathsf{T}}(M\bar{u})$$

Thus, if  $\hat{z}, \hat{w} \geq 0$  then  $\hat{z} \in SOL(q, M)$ . Otherwise, since  $\bar{u}_{\alpha} > 0$  and  $\bar{v}_{\bar{\alpha}} > 0$ , for sufficiently large  $\lambda > 0$  we have that

$$\begin{bmatrix} \hat{z}_{\alpha} \\ \hat{w}_{\bar{\alpha}} \end{bmatrix} + \lambda \begin{bmatrix} \bar{u}_{\alpha} \\ \bar{v}_{\bar{\alpha}} \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Letting  $\bar{\lambda}$  be the smallest  $\lambda$  satisfying the inequality above, and setting  $\bar{z}_0 = \bar{\lambda}, \bar{z} = \hat{z} + \bar{\lambda}\bar{u}$  we get that  $(\bar{z}_0, \bar{z}, 1, \bar{u}) \in SR_1(e, q, M)$ . Note that constructing  $(\bar{z}_0, \bar{z})$  whenever a nondegenerate

#### **Extensions** 9

In this section we show how to extend our reductions whenever a general positive covering vector is used rather than e. The key to the results in this section is the following proposition.

**Proposition 6** Given  $d \in \mathcal{R}_{++}^m$ , let D be the diagonal matrix with  $D_{ii} = d_i$ . Then,  $(\bar{z}_0, \bar{z}) \in ELCP(d, q, M)$  if and only if  $(\bar{z}_0, D\bar{z}) \in ELCP(e, D^{-1}q, D^{-1}MD^{-1})$ .

Proof. The proposition is easily verified by observing that

 $q + dz_0 + Mz \ge 0, \ z \ge 0$  if and only if  $D^{-1}(q + dz_0 + MD^{-1}Dz) \ge 0, \ Dz \ge 0,$ 

and  $z^{\intercal}(q + dz_0 + Mz) = 0$  if and only if  $z^{\intercal}DD^{-1}(q + dz_0 + MD^{-1}Dz) = 0$ . 

**Corollary 7** Given  $d \in \mathcal{R}_{++}^m$ , let D be the diagonal matrix with  $D_{ii} = d_i$ . Then,

(i)  $\overline{z} \in LCP(q, M)$  if and only if  $D\overline{z} \in LCP(D^{-1}q, D^{-1}MD^{-1})$ .

(ii)  $M \in \mathbf{R}(d)$  if and only if  $D^{-1}MD^{-1} \in \mathbf{R}(e)$ .

(iii)  $M \in \mathbf{R}_{\mathbf{0}}(d)$  if and only if  $D^{-1}MD^{-1} \in \mathbf{R}_{\mathbf{0}}(e)$ .

#### Proof.

- (i) Results from Proposition 6 by considering  $\bar{z}_0 = 0$ .
- (ii)-(ii) Result directly from (i) and the definitions of  $\mathbf{R}(d)$  and  $\mathbf{R}_{\mathbf{0}}(d)$ .

By Proposition 6 and Corollary 7 it can be readily verified that the results of Section 6 can be extended to  $\mathbf{R}(d) - LCP(q, M)$  by considering  $\mathbf{R}(e) - LCP(D^{-1}q, D^{-1}MD^{-1})$ . Similarly, the results of Section 7 can be extended to  $\mathbf{R}_{0}(d) - LLCP(d, q, M)$  by considering  $\mathbf{R}_{0}(e) - LLCP(e, D^{-1}q, D^{-1}MD^{-1})$ . Finally, given that LCP(d, M) is nondegenerate we can (by Proposition 6, Corollary 7, and the results of sections 7 and 8) reduce any LLCP(d, q, M) where LCP(d, M) is nondegenerate to a bimatrix game whose cost matrix is

$$C = \begin{bmatrix} 1 & -e^{\mathsf{T}} & -\beta \prod_{i=1}^{m} d_i + 1 \\ e & D^{-1} M D^{-1} & D^{-1} q + e \\ 0 & 0 & 1 \end{bmatrix}.$$

Let  $\begin{bmatrix} \bar{z}_0 \\ \bar{z} \\ \bar{t} \end{bmatrix} \in SNE(C)$ . Based on Proposition 6 and Corollary 7 we replace  $\bar{z}$  with  $D^{-1}\bar{z}$  and proceed to recover a 'solution' to LLCP(d.q.M) by following the steps prescribed in sections 7 and 8.

### 10 Concluding remarks

- 1. The main result of this paper is that for almost any given  $d \in \mathcal{R}_{++}^m, q \in \mathcal{R}^m, M \in \mathcal{R}^{m \times m}$ , it is possible to effectively set up a symmetric bimatrix game whose Nash equilibria correspond one-to-one to all the endpoints (excluding the one corresponding to the primary ray) of the directed graph associated with LCP(q, M) and the Lemke method with a covering vector d. The only condition is that any Nash equilibrium corresponding to a type 1 secondary direction (that is a solution for LCP(d, M)), has to be nondegenerate. Note that if this is not the case, we can perturb (by standard LP techniques) d to a  $\hat{d}$  which, when used as the covering vector, will guarantee that the reduction will work. This observation also means that for any given M, q the reduction is workable for all covering vectors  $d \in \mathcal{R}_{++}^m$ , with the exception of a finite number of sets of measure 0.
- 2. As a consequence of the main result as specified above, the reduction will resolve any LCP(q, M) which is Lemke(d)  $\mathcal{PPAD}$ -verified. Note that all the major matrix classes of M which are known to be Lemke(d) resolvable for all q are actually known to be Lemke(d)  $\mathcal{PPAD}$ -verified (and typically for all  $d \in \mathcal{R}^m_{++}$ ). These classes (which are all subsets of  $\mathbf{USR}(d)$ ) and their relationships are depicted in Figure 1.
- 3. The direct reductions which are presented in this paper highlight the importance of the problem of 2-NASH within mathematical programming. In a sense, we show that as any LP can be directly reduced to zero-sum game (see [Dan51] and [Adl12]), it is analogously possible to directly reduce any LCP(q, M) which is Lemke(d)  $\mathcal{PPAD}$ -verified to a 2-NASH problem, thus showing that many of the results regarding such problems are relevant to LCP theory.
- 4. The reductions in sections 6, 7, and 8 are simple and easy to execute. Thus, any algorithm that is applicable to bimatrix games can be directly used to solve instances of  $\mathbf{RSU}(d) - LCP(q, M)$ and LLCP(d, q, M). Also, considering that the proposed reductions are bijections, algorithms with a variety of goals, such as enumerating all, or specific subsets of Nash equilibrium points can be applied for similar goals regarding the solutions of linear complementarity problems for which our reductions are applicable. It should be noted that there is a vast literature covering the subjects of computing and enumerating Nash equilibria of bimatrix games (see e.g. the surveys in [vSt02],[vSt07] and the papers introduced in [vSt10]).
- 5. Over the years several refinements of Nash equilibrium have been introduced. In particular, some results regarding the existence and computation of these refinements have been established. In [MT98] some of these refinements are generalized to LCPs. The reduction of LCP(q, M) which are Lemke(d)  $\mathcal{PPAD}$ -verified (e.g. where  $M \in \mathbf{USR}(d)$ ) to symmetric NASH-2, provides us with a tool to investigate analogous questions with respect to the generalized refinements to such LCPs. For example, in a forthcoming paper, we demonstrate such an analysis by proving that any LCP(q, M) with  $M \in \mathbf{R}(e)$  has a proper solution. As a corollary of this analysis we prove that the (unique) solution of LCP(q, M) where  $M \in \mathbf{P}$  is proper and thus settle a conjecture posed in [MT98] (where it is proved for  $2 \times 2$  matrices).
- 6. The simple reductions proposed in this paper allow us, whenever applicable, to potentially gain additional insight into the nature of models represented by these LCPs. This seems to be especially useful for economic models such as market equilibrium.

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