Sufficient matrices belong to L

We dedicate this article to the memory of George B. Dantzig and Carlton E. Lemke who continue to inspire researchers everywhere.

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Abstract. In this paper, we establish a significant matrix class inclusion that seems to have been overlooked in the literature of the linear complementarity problem. We show that $P^*$, the class of sufficient matrices, is a subclass of $L$. In the course of demonstrating this inclusion, we introduce other new matrix classes that forge interesting new connections between known matrix classes.

1. Introduction

Matrix classes have always played a prominent role in the study of the Linear Complementarity Problem (LCP): Given $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^{n \times 1}$, find a vector $x \in \mathbb{R}^n$ such that

\[
\begin{align*}
x & \geq 0, \\
q + Mx & \geq 0, \\
x^T(q + Mx) & = 0.
\end{align*}
\]

We denote this system by the pair $(q, M)$. Its feasible set (the vectors that satisfy the two sets of linear inequalities) is denoted $FEA(q, M)$, whereas its solution set (the set of feasible vectors satisfying the third ("complementarity") condition) is denoted $SOL(q, M)$. Abundant coverage of matrix classes in the LCP is available in the monographs [6], [24], and the research articles in the reference list below.

The present study began as an effort to answer a question in the theory of the LCP that seems not to have been posed before: Is the class $P^*$ of sufficient matrices included in the class $L$? We answer this question in the affirmative. Furthermore, we explore the connections of $L$ with other matrix classes. In so doing, we establish a bridge between copositive matrices and certain subclasses of $P_0$.

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As originally defined in the literature [7], the class of “sufficient matrices” mentioned above was denoted SU. It was conjectured and partially proved in [17] that SU and the class \( P^* \) defined by Kojima, Megiddo, Noma, and Yoshise [19] are actually the same. The latter class equals the union of \( P^*(\kappa) \) where \( \kappa \) is a non-negative number. A real \( n \times n \) matrix \( M \) belongs to \( P^*(\kappa) \) if and only if

\[
(1 + 4\kappa) \sum_{i \in I_+(x)} x_i (Mx)_i + \sum_{i \in I_-(x)} x_i (Mx)_i \geq 0 \quad \forall x \in \mathbb{R}^n
\]

and

\[
I_+(x) = \{ i : x_i (Mx)_i > 0 \} \quad \text{and} \quad I_-(x) = \{ i : x_i (Mx)_i < 0 \}.
\]

The aforementioned conjecture was established conclusively by Váliaho [25]. Hereafter, we adopt the notation \( P^* \) for the class of sufficient matrices. We observe that the basic definition of \( P^* \) does not suggest a finite test for membership in that class. Nevertheless, by virtue of the equivalence of \( P^* \) and SU, such a test is available. See [5].

It is known that \( P^* \) contains several other classes of interest in the study of the LCP. Among these are: the positive semidefinite matrices, the matrices whose principal minors are all positive, the matrices having only one principal minor of value zero and all the rest positive, and the adequate matrices. Interestingly, all these subclasses of \( P^* \) are known to be subclasses of the class \( L \) which was introduced by Eaves in [11] where many of these inclusions were first proved. They contributed directly to our hunch that \( P^* \subset L \). It should be noted that \( L \subset Q_0 \), meaning that whenever \((q, M)\) is feasible, it is solvable. Indeed, Eaves showed that for \( M \in L \), Lemke’s algorithm [20] will process \((q, M)\), that is, find a solution or give evidence that the problem is infeasible.

2. Notation and terminology

In this section we set down a bit of notation and give a few standard definitions to get things started. Others will be given later, as needed. Our aim, for the moment, is to enable the reader to understand the material up to this point.

All non-indexed matrices and vectors are in \( \mathbb{R}^{n \times n} \) and \( \mathbb{R}^n \), respectively, but we write \((x_1, \ldots, x_n)\) instead of \([x_1 \cdots x_n]^T\). If \( A \) is a matrix, \( A_{\alpha} \) denotes the \( i \)th column of \( A \). Whenever \( \alpha \subseteq \{1, 2, \ldots, n\} \) we denote by \( \bar{\alpha} \) the complement of \( \alpha \) with respect to \( \{1, 2, \ldots, n\} \).

**Definition 2.1.** A matrix \( M \in E_0 \) if for every \( 0 \neq x \geq 0 \) there exists some \( i \) such that \( x_i > 0 \) and \((Mx)_i \geq 0 \). Introduced in [11] and [18], the matrices in this class are said to be semimonotone. See [6, 3.13.18]

**Definition 2.2.** A matrix \( M \in E_1 \) if for every nonzero \( x \in \text{SOL}(0, M) \) there exists nonnegative diagonal matrices \( \Gamma \) and \( \Omega \) such that \((\Gamma M + M^T \Omega)x = 0 \) and \( \Omega x \neq 0 \).

**Definition 2.3.** \( L = E_0 \cap E_1 \),

**Definition 2.4.** A matrix \( M \in CSU \) (is Column Sufficient) if for all \( x \)

\[
x_i (Mx)_i \leq 0 \quad (i = 1, 2, \ldots, n) \quad \text{implies that} \quad x_i (Mx)_i = 0 \quad (i = 1, 2, \ldots, n).
\]
Definition 2.5. A matrix $M \in \text{RSU}$ (is Row Sufficient) if $M^T \in \text{CSU}$.

Definition 2.6. $P_* = \text{CSU} \cap \text{RSU}$. The matrices in this class are said to be sufficient.

Remark. It is known [7, p. 238] that RSU (and hence CSU) is a subclass of $P_0$, the class of matrices with nonnegative principal minors. This fact makes RSU and CSU subclasses of $E_0$ as we know from [11].

Definition 2.7. Given a vector $x$, we shall denote by $\sigma(x)$ the support of $x$, i.e. $\{ i : x_i \neq 0 \}$.

For a given matrix $M \in \mathcal{R}^{n \times n}$ and a given partition $\alpha, \bar{\alpha}$ of $\{1, 2, \ldots, n\}$, $C_M(\alpha)$ denotes the $n \times n$ matrix for which

$$
[C_M(\alpha)]_{ij} = \begin{cases} -M_{ij} & i \in \alpha \\ I_{ij} & i \in \bar{\alpha} \end{cases}
$$

(2.1)

With a slight abuse of language, the matrices $C_M(\alpha)$ are called complementary submatrices (of $[-M I]$). The sets

$$
pos C_M(\alpha) = \{ C_M(\alpha)x | x \geq 0 \}
$$

are the complementary cones, relative to $M$. The union over $\alpha$ of these cones is denoted $K(M)$ and called the complementary range of $M$. Clearly, $\text{SOL}(q, M) \neq \emptyset$ if and only if $q \in K(M)$.

It is customary to simplify the notation $C_M(\alpha)$ to $C_\alpha$ when the matrix $M$ is understood. Further simplification is achieved (and usually not restrictive) when $\alpha$ is a leading subset of $\{1, 2, \ldots, n\}$, in which case we have

$$
C_\alpha = \begin{bmatrix} -M_{\alpha\alpha} & 0 \\ -M_{\bar{\alpha}\alpha} & I \end{bmatrix}.
$$

(2.2)

This matrix is singular if and only if the principal submatrix $M_{\alpha\alpha}$ of $M$ is singular. When this is the case, the corresponding complementary cone $pos C_\alpha$ is said to be degenerate and nondegenerate otherwise. A strongly degenerate complementary cone $pos C_\alpha$ is one for which

$$
\{ x | C_M(\alpha)x = 0, \ 0 \neq x \geq 0 \} \neq \emptyset.
$$

Strong degeneracy is intimately connected with a criterion for the boundedness of the solution set of an LCP, a subject that has been considered in several publications, such as [2], [6], [8], [9], [10] [15] and [21]. Suppose $\text{SOL}(q, M)$ is nonempty. Then there exist a positive, but finite, number of index sets $\alpha$ such that $q \in pos C_M(\alpha)$. The elements of $\text{SOL}(q, M)$ come from the solution sets $X(\alpha)$ of the corresponding systems

$$
C_M(\alpha)x = q, \ \ x \geq 0.
$$

By a well known theorem of linear inequality theory (see for instance [13]) the set $X(\alpha)$ is unbounded if and only if (in our terminology) the complementary cone $pos C_\alpha$ is strongly degenerate.
The matrix $M$ belongs to $Q_0$ when for all $q$
\[ \text{FEA}(q, M) \neq \emptyset \implies \text{SOL}(q, M) \neq \emptyset. \]
It was shown by Eaves [11] that $M \in Q_0$ if and only if $K(M)$ is convex. This result is also covered in [6, 3.2.1].

It was mentioned in Section 1, that the class $L$ is a subclass of $Q_0$. In fact, this is also true of the class RSU as first proved in [7]. It was noted there – and proved in [1] – that since $\text{RSU} \subset \text{P}_0 \cap Q_0$, Lemke’s algorithm [20] will process any LCP $(q, M)$ with a row sufficient matrix $M$. Furthermore, it was shown in [3] that the Principal Pivoting Method of Cottle and Dantzig also processes this class of linear complementarity problems.

3. Preliminary results

We start by proving a series of lemmas which will be used to show that the class $P_\ast$ of sufficient matrices is a subclass of $L$. Since $L = E_0 \cap E_1$, and it is known [7] that $P_\ast \subset P_0 \subset E_0$, the task boils down to proving $P_\ast \subset E_1$.

Remark. We establish the following three lemmas for a new matrix class even larger than RSU (which, in turn, is even larger than $P_\ast$). We believe these easily proved results are of independent interest because they seem to embrace both RSU and $C_0$, the latter being the class of all copositive matrices, i.e., those for which
\[ x \geq 0 \implies x^T M x \geq 0. \]
It will be helpful to define the polyhedral cone
\[ T(M) = \{ y \mid y \geq 0, y^T M \leq 0 \}. \]
(This cone appears in Gowda’s paper [14] where copositive star matrices were introduced. See also [6, 3.8.13].) Note that for any square matrix $M$, we have
\[ \text{SOL}(0, -M^T) \subseteq \text{FEA}(0, -M^T) = T(M). \]

Definition 3.1. Let $T_\ast$ denote the class of all square matrices $M$ such that
\[ T(M) = \text{SOL}(0, -M^T). \]

Lemma 3.1. $\text{RSU} \subset T_\ast$.

Proof. Let $M \in \text{RSU}$. We need only prove $T(M) \subseteq \text{SOL}(0, -M^T)$. For all $y \in T(M)$, we have $y_i^T (M^T y)_i \leq 0$ $(i = 1, 2, \ldots, n)$). However, $M \in \text{RSU}$, so $M^T \in \text{CSU}$, which implies that $y_i^T (M^T y)_i = 0$ $(i = 1, 2, \ldots, n)$. Thus we can conclude that $T(M) \subseteq \text{SOL}(0, -M^T)$. Hence $M \in T_\ast$. $\Box$

Remark. The class $T_\ast$ contains more than just RSU. For example, it contains $C_0$. Indeed, let $M$ be copositive. Then by the above, $\text{SOL}(0, -M^T) \subseteq T(M)$. Now if $y \in T(M)$, then $y^T M y \leq 0$ with $y \geq 0$ so that $y^T M y = 0$; hence $T(M) \subseteq \text{SOL}(0, -M^T)$. In short, $C_0 \subset T_\ast$ (strictly).
Lemma 3.2. Suppose $M \in T_*$. If $y, z \in \text{SOL}(0, -M^T)$, then $y_i(M^T z)_i = 0$ ($i = 1, 2, \ldots, n$) and $z_i(M^T y)_i = 0$ ($i = 1, 2, \ldots, n$).

Proof. Suppose $y, z \in \text{SOL}(0, -M^T)$. Since by Definition 3.1, $\text{SOL}(0, -M^T)$ is a convex cone, we have $\frac{1}{2}y + \frac{1}{2}z \in \text{SOL}(0, -M^T)$. Thus,

$$0 = \left(\frac{1}{2}y + \frac{1}{2}z\right)^T (-M^T) \left(\frac{1}{2}y + \frac{1}{2}z\right) = \frac{1}{4}(y^T (-M^T)y + z^T (-M^T)z - y^T M^T z - z^T M^T y) = -\frac{1}{4}(y^T M^T z + z^T M^T y).$$

Noticing that $y \geq 0$, $z \geq 0$, $M^T y \leq 0$, $M^T z \leq 0$, completes the proof. \qed

Lemma 3.3. Suppose $M \in T_*$ and $T(M) \neq \{0\}$. There exists $\beta$, a nonempty subset of $\{1, 2, \ldots, n\}$, such that for every $y \in T(M)$, $(M^T y)_\beta = 0$ and, when $\bar{\beta}$ is nonempty, $y_{\bar{\beta}} = 0$.

Proof. Let $i \in \beta$ if, and only if, there exists $z \in T(M)$ such that $z_i > 0$; that is to say,

$$\beta = \bigcup_{z \in T(M) \setminus \{0\}} \sigma(z).$$

The index set $\beta$ is nonempty; moreover, since $T(M)$ is a convex cone, $\beta$ is, in fact, the support of some vector $\hat{z} \in T(M)$. Thus, by Lemma 3.2, if $y \in T(M)$, then $y^T \beta = 0$, and obviously for nonempty $\bar{\beta}$, $y_{\bar{\beta}} = 0$. \qed

At this juncture, we verify an assertion which, in essence, was made long ago by Garcia [12].

Proposition 3.1. If $M \in R^{n \times n}$, then $M \in E_1$ if and only if for every $z \in \text{SOL}(0, M) \setminus \{0\}$, there exists $y$ such that:

$$y \in T(M) \setminus \{0\} \quad \sigma(y) \subseteq \sigma(z) \quad \sigma(M^T y) \subseteq \sigma(Mz).$$

Proof. Suppose $M \in E_1$. If $z \in \text{SOL}(0, M) \setminus \{0\}$ there exist nonnegative diagonal matrices $\Gamma$ and $\Omega$ such that $\Omega z \neq 0$ and $(\Gamma M + M^T \Omega)z = 0$. Now if $\text{SOL}(0, M) = \{0\}$, there is nothing to prove. Let $z \in \text{SOL}(0, M) \setminus \{0\}$. Define $y = \Omega z$. Then $y$ is nonnegative and nonzero. It is then clear that $M^T y \leq 0$, so that (3.1) holds. Furthermore, if $z_i = 0$, then $y_i = 0$. Hence (3.2) holds. Next, if $(Mz)_i = 0$, then $(M^T y)_i = 0$, so (3.3) holds.

Conversely, if $\text{SOL}(0, M) = \{0\}$, there is nothing to prove. Hence suppose $z \in \text{SOL}(0, M) \setminus \{0\}$ and that there exists a nonzero vector $y$ such that (3.1)–(3.3) hold. By (3.2), it is clear that there exists a nonnegative diagonal matrix $\Omega$ such that $y = \Omega z$. By (3.3), there exists a nonnegative diagonal matrix $\Gamma$ such that $\Gamma M z = -M^T y$. Accordingly, $M \in E_1$. \qed
Lemma 3.4. For a given $M \in \mathbb{R}^{n \times n}$ let $\alpha, \bar{\alpha}$ be a partition of $\{1, 2, \ldots, n\}$. Then there exists a vector $y$ such that

$$
y^T M \leq 0, \quad 0 \neq y \geq 0
$$

$$
y^T M_{\alpha \alpha} = 0
$$

$$
y_{\bar{\alpha}} = 0
$$

if and only if there is no solution of the system

$$
C_\alpha x = -Mr + s
$$

$$
x \geq 0, \quad r \geq 0, \quad s > 0.
$$

(3.4)

Proof. It is clear that (3.4) is equivalent to the system

$$
C_\alpha x + Mr - s = 0
$$

$$
s - t = d \text{ (where } d \text{ is an arbitrary positive vector)}
$$

(3.5)

$$
x \geq 0, \quad r \geq 0, \quad s \geq 0, \quad t \geq 0.
$$

By Farkas’s lemma (see, e.g., [6, p. 109]), (3.5) has no solution if and only if there exist $y$ and $u$ such that

$$
y^T C_\alpha \leq 0, \quad y^T M \leq 0, \quad y - u \geq 0, \quad 0 \neq u \geq 0,
$$

which, by setting $y = (y_\alpha, y_{\bar{\alpha}})$ and using (2.1), completes the proof. \(\Box\)

Corollary 3.1. For a given $M \in \mathbb{R}^{n \times n}$, and a partition $\alpha, \bar{\alpha}$ of $\{1, 2, \ldots, n\}$ there exists $y$ such that

$$
y \in T(M) \setminus \{0\}
$$

$$
\sigma(y) \subseteq \alpha
$$

(3.6)

$$
\sigma(M^T y) \subseteq \bar{\alpha}
$$

(3.7)

(3.8)

if and only if $\text{int pos}[-M, I] \cap \text{pos} C_\alpha = \emptyset$.

Proof. The assertion follows directly from Lemma 3.4 and the observation that $q \in \text{int pos}[-M, I]$ if and only if there exist $r \geq 0, s > 0$ such that $q = -Mr + s$. \(\Box\)

Definition 3.2. A matrix $M$ belongs to $R_1$ if $\text{SOL}(q, M)$ is bounded for all $q \in \text{int pos}[-M, I]$.

Remark. The class $R_1$ contains the class $R_0$. The latter class was introduced in [12] by Garcia who denoted it by $E^*(0)$. The class $R_0$ is discussed in [6, 3.9.23] where it is shown that such matrices $M$ are characterized by the boundedness of the solution sets for all LCPs $(q, M)$, not just those for which $q \in \text{int pos}[-M, I]$ as in the case of $R_1$-matrices.

Taking account of the criterion for unboundedness of the solution set of an LCP (stated above), we have
Lemma 3.5. \( M \in R_1 \) if and only if for every \( \alpha \subseteq \{1, 2, \ldots, n\} \)

\[
\{0 \neq x \geq 0 \mid C_\alpha x = 0\} \neq \emptyset \Rightarrow \text{int pos}[-M, I] \cap \text{pos} C_\alpha = \emptyset. \quad \square
\]

We now return to \( T_* \) and give a criterion for such a matrix to belong to the class \( E_1 \). This will lead to a criterion for membership in \( L \).

Proposition 3.2. \( T_* \cap R_1 \subset T_* \cap E_1 \).

Proof. Suppose \( z \in \text{SOL}(0, M) \setminus \{0\} \) and let \( \sigma_1 = \sigma(z) \), \( \sigma_2 = \sigma(Mz) \) and \( \sigma_3 \) be the complement of \( \sigma_1 \cup \sigma_2 \) with respect to \( \{1, 2, \ldots, n\} \). Define \( \alpha = \sigma_1 \cup \{i \in \sigma_3 \mid i \in \beta\} \) and its complement \( \bar{\alpha} = \sigma_2 \cup \{i \in \sigma_3 \mid i \in \beta\} \) where \( \beta, \bar{\beta} \) are defined as in Lemma 3.3 and \( T(M) \neq \{0\} \). In the event that \( T(M) = \{0\} \), we take \( \beta = \emptyset \). For simplicity, we assume that \( \alpha \) is the leading index set within \( \{1, 2, \ldots, n\} \). Let \( x = (z_\alpha, (Mz)_{\bar{\alpha}}) \).

\[
\text{C}_\alpha x = 0, \quad 0 \neq x \geq 0, \quad \text{where } \text{C}_\alpha \text{ is defined as in (2.2). (3.9)}
\]

Suppose \( M \in R_1 \). Then (3.9) and Lemma 3.5 imply that \( \text{int pos}[-M, I] \cap \text{pos} C_\alpha = \emptyset \), so by Corollary 3.1 there exists \( y \) satisfying (3.6)–(3.8).

Suppose \( z_i = 0 \) for some \( i \in \alpha \). Then, \( i \in \beta \) by the definition of \( \alpha \), so by Lemma 3.3, \( y_i = 0 \). Noticing that \( y_{\bar{\alpha}} = 0 \), we conclude that \( \sigma(y) \subseteq \sigma(z) \).

Suppose \( (Mz)_i = 0 \) for some \( i \in \bar{\alpha} \). Then \( i \in \beta \) by the definition of \( \bar{\alpha} \). Thus, by Lemma 3.3, \( (M^T y)_i = 0 \). Noticing that \( (M^T y)_\alpha = 0 \), we conclude that \( \sigma (M^T y) \subseteq \sigma (Mz) \).

From view of Proposition 3.1 we have shown that \( M \in E_1 \). \( \square \)

Using Proposition 3.2, and recalling that \( L = E_0 \cap E_1 \) and \( RSU \subset E_0 \cap T_* \), we can immediately state the following corollary which sets up a sufficient condition for a matrix \( M \in RSU \) to be in \( L \).

Corollary 3.2. Suppose \( M \in RSU \). If \( M \in R_1 \), then \( M \in L \).

4. The main results

We are now in a position to establish our main results.

Theorem 4.1. \( \text{CSU} \cap Q_0 \subset R_1 \).

Proof. Suppose \( M \) is a matrix belonging to \( \text{CSU} \cap Q_0 \) but not to \( R_1 \). Then in view of Lemma 3.5 it must be the case that for some element \( \tilde{q} \in \text{int pos}[-M \mid I] \), we have vectors \( z \) and \( z^\lambda = \tilde{z} + \lambda z \) such that \( z \in \text{SOL}(0, M) \setminus \{0\} \) and \( z^\lambda \in \text{SOL} (\tilde{q}, M) \) for all \( \lambda \geq 0 \). Let

\[
\alpha = \sigma(z), \quad \beta = \sigma(z + \tilde{z}) \setminus \alpha, \quad \gamma = \alpha \cup \beta.
\]

Next choose \( p \in K(M) \) such that

\[
\begin{align*}
p_i - \tilde{q}_i &< 0 & \text{if } i \in \alpha \\
p_i - \tilde{q}_i &> 0 & \text{if } i \in \gamma.
\end{align*}
\]

\[
p_i - \tilde{q}_i = 0 & \text{ if } i \in \beta
\]

Suppose \( \tilde{q} \in \text{int pos}[-M \mid I] \), we have vectors \( z \) and \( z^\lambda = \tilde{z} + \lambda z \) such that \( z \in \text{SOL}(0, M) \setminus \{0\} \) and \( z^\lambda \in \text{SOL}(\tilde{q}, M) \) for all \( \lambda \geq 0 \). Let

\[
\alpha = \sigma(z), \quad \beta = \sigma(z + \tilde{z}) \setminus \alpha, \quad \gamma = \alpha \cup \beta.
\]

Next choose \( p \in K(M) \) such that

\[
\begin{align*}
p_i - \tilde{q}_i &< 0 & \text{if } i \in \alpha \\
p_i - \tilde{q}_i &> 0 & \text{if } i \in \gamma.
\end{align*}
\]

\[
p_i - \tilde{q}_i = 0 & \text{ if } i \in \beta
\]
Such a vector \( p \) is guaranteed to exist since \( \tilde{q} \in \text{int post}[-M I] \) and \( M \in Q_{0} \).

Choose any \( w \in\text{SOL}(p, M) \). The remainder of the argument will show that for sufficiently large \( \lambda \) the vector \( z^{\lambda} - w \) violates the column sufficiency property of \( M \).

Note that for all such \( \lambda \), we have \( (z^{\lambda} - w)_{\alpha} > 0 \). Consequently we have

\[
(M(z^{\lambda} - w))_{\alpha} = -\tilde{q}_{\alpha} - (Mw)_{\alpha} \leq -\tilde{q}_{\alpha} + p_{\alpha} < 0.
\]

Thus, we have shown

\[
(z^{\lambda} - w)_{\alpha}(M(z^{\lambda} - w))_{\alpha} < 0.
\]

Now let \( i \in \beta \). Then \( \tilde{z}_{i} > 0 \) and \( z_{i} = 0 \). For such \( i \) we also have \( p_{i} = \tilde{q}_{i} \). For all these \( i \) we have

\[
(z^{\lambda} - w)_{i}(M(z^{\lambda} - w))_{i} = (z^{\lambda} - w)_{i}((Mz^{\lambda})_{i} - (Mw)_{i} + \tilde{q}_{i} - p_{i})
\]

\[
= z_{i}^{\lambda}((Mz^{\lambda})_{i} + \tilde{q}_{i}) - w_{i}((Mz^{\lambda})_{i} + \tilde{q}_{i})
\]

\[
+ z_{i}^{\lambda}(-(Mw)_{i} - p_{i}) - w_{i}(-(Mw)_{i} - p_{i})
\]

\[
\leq 0.
\]

Finally, let \( i \in \gamma \). In this case, we must have \( z_{i} = \tilde{z}_{i} = z_{i}^{\lambda} = 0 \). From this we obtain

\[
(z^{\lambda} - w)_{i}(M(z^{\lambda} - w))_{i} = -w_{i}((Mz^{\lambda})_{i} - (Mw)_{i})
\]

\[
\leq -w_{i}((Mz^{\lambda})_{i} - (Mw)_{i} + \tilde{q}_{i} - p_{i})
\]

\[
= -w_{i}(Mz^{\lambda} + \tilde{q})_{i} + w_{i}(Mw + p)_{i}
\]

\[
= -w_{i}(Mz^{\lambda} + \tilde{q})_{i}
\]

\[
\leq 0.
\]

Thus, we have shown that for sufficiently large \( \lambda \)

\[
(z^{\lambda} - w)_{i}(M(z^{\lambda} - w))_{i} \leq 0 \quad \text{for all } i = 1, \ldots, n;
\]

but since \( \alpha \neq \emptyset \), it is not the case that for sufficiently large \( \lambda \)

\[
(z^{\lambda} - w)_{i}(M(z^{\lambda} - w))_{i} = 0 \quad \text{for all } i = 1, \ldots, n.
\]

Since \( M \) is column sufficient, this is a contradiction. Hence \( M \in R_{1} \).

\( \square \)

This brings us to the first of the main results we set out to prove.

\textbf{Corollary 4.1.} Every sufficient matrix belongs to the class \( L \).

\textit{Proof.} Suppose that \( M \in P_{a} \). Recall that \( P_{a} = \text{CSU} \cap \text{RSU} \subset \text{RSU} \subset Q_{0} \). Accordingly, \( P_{a} \subset \text{CSU} \cap Q_{0} \), so by Theorem 4.1, \( M \in R_{1} \). Thus, in view of Corollary 3.2, we can conclude that \( M \in L \).

\( \square \)

The reverse of Proposition 3.2—without the restriction that \( M \in T_{a} \)—follows from a result of Gowda and Sznajder [16, Theorem 11]. Here we prove it afresh and thereby avoid developing their conceptual apparatus.
Proposition 4.1. \( E_1 \subset R_1 \).

Proof. Suppose \( M \in E_1 \) and let \( \alpha, \bar{\alpha} \) be a partition of \( \{1, 2, \ldots, n\} \) where for simplicity, \( \alpha \) is written as a leading index set within \( \{1, 2, \ldots, n\} \). Suppose there exists \( x \) such that
\[
C_{\alpha} x = 0, \quad 0 \neq x \geq 0
\]
where \( C_{\alpha} \) is defined as in (2.2).

Let \( z = (z_{\alpha}, z_{\bar{\alpha}}) = (x_{\alpha}, 0) \), so \( Mz = ((Mz)_{\alpha}, (Mz)_{\bar{\alpha}}) = (0, x_{\bar{\alpha}}) \) and \( z \in \text{SOL}(0, M) \) \{0\} with \( \sigma(z) \subseteq \alpha \) and \( \sigma(Mz) \subseteq \bar{\alpha} \). By Proposition 3.1 there exists \( y \) satisfying (3.1)–(3.3). Obviously \( y \) satisfies (3.6)–(3.8) as well, so by Corollary 3.1, \( \text{int pos}[-M \ I] \cap \text{pos} C_{\alpha} = \emptyset \), which, in view of Lemma 3.5, implies that \( M \in R_1 \).

Combining Propositions 3.2 and 4.1, and recalling that \( L = E_0 \cap E_1 \), we get a complete characterization of the class \( L \) within \( E_0 \cap T_* \).

Corollary 4.2. If \( Y \) is a subclass of \( E_0 \cap T_* \), then
\[
Y \cap R_1 = Y \cap L.
\]

The significance of this corollary for the classes RSU and \( C_0 \) is obvious. Interestingly, the characterization above can be alternatively expressed in terms of a known (see, for example, [2] and [6, 6.3.14]) class of matrices which is defined below.\(^1\)

Definition 4.1. A matrix \( M \) belongs to \( R_2 \) if \( \text{SOL}(q, M) \) is bounded for all \( q \in \text{int} K(M) \).

It is clear that \( R_1 \subset R_2 \), but more can be said.

Proposition 4.2. Suppose \( M \in E_0 \cap T_* \). Then \( M \in R_1 \) if and only if \( M \in R_2 \cap Q_0 \).

Proof. Suppose \( M \in R_2 \cap Q_0 \). Recall that \( M \in Q_0 \) means that \( K(M) = \text{pos}[-M \ I] \), which, by definition, implies that \( M \in R_1 \).

Suppose \( M \in R_1 \). From the assumption \( M \in E_0 \cap T_* \) and Proposition 3.3, we see that \( M \in E_0 \cap E_1 = L \subset Q_0 \). Thus, since \( R_1 \subset R_2 \), we conclude that \( M \in R_2 \cap Q_0 \).

This, in light of Corollary 4.2, leads to a characterization of the class \( L \) within \( E_0 \cap T_* \) in terms of class \( R_2 \).

Corollary 4.3. If \( Y \) is a subclass of \( E_0 \cap T_* \), then
\[
Y \cap R_2 \cap Q_0 = Y \cap L.
\]

Remark. At this time, it is an open question as to whether there exist row sufficient matrices that do not belong to \( L \). Such an example would be needed to demonstrate that restricting RSU to be in \( R_1 \) (or, equivalently, in \( R_2 \cap Q_0 \)) is required to make it a subclass of \( L \). On the other hand, an example such as
\[
M = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}
\]
\(^1\) The notation \( R_2 \) for this class of matrices appears for the first time in this paper.
shows that $P_0 \cap Q_0$ is not a subclass of $L$. Furthermore, Murthy, Parthasarathy, and Ravindran [23] show that the matrix

$$M = \begin{bmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & -1 & 1 & 0 \\
-1 & 1 & 0 & 1
\end{bmatrix}$$

belongs to $C_0 \cap Q$ but not to $R_0$. It is elementary to show that it does not belong to $E_1$ (and therefore to $L$) either.

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References